

# Almost Sure Stabilization for Feedback Controls of Regime-switching Linear Systems with A Hidden Markov Chain

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**Abstract**—This work is devoted to the almost sure stabilization of adaptive control systems that involve an unknown Markov chain. The control system displays continuous dynamics represented by differential equations and discrete events given by a hidden Markov chain. In the previous investigation on this class of problems, averaging criteria were used, which provides only the system behavior in some expectation sense. A closer scrutiny of the system behavior necessarily requires the consideration of sample path properties. Different from previous work on stabilization of adaptive controlled systems with a hidden Markov chain, where average criteria were considered, this work focuses on the almost sure stabilization or sample path stabilization of the underlying processes. Under simple conditions, it is shown that as long as the feedback controls have linear growth in the continuous component, the resulting process is regular. Moreover, by appropriate choice of the Lyapunov functions, it is shown that the adaptive system is stabilizable almost surely. As a by-product, it is also established that the controlled process is positive recurrent.

**Index Terms**—Adaptive control, hidden Markov chain, almost sure stabilization.

## 1. INTRODUCTION

THIS work deals with almost sure stabilization of a class of adaptive control systems in continuous-time with an unknown parameter process that is a hidden Markov chain. The systems belong to the class of partially observed control systems. Naturally, one estimates the parameter process by using nonlinear filtering techniques and then uses the estimator in the systems in order to design adaptive control strategies. The motivation of our study stems from consideration of the following problem. Consider the process

$$\dot{X}(t) = A_{\alpha(t)}X(t) + B_{\alpha(t)}U(t)$$

where  $\alpha(t)$  is a continuous-time Markov chain taking values in a finite set  $\mathcal{M} = \{1, \dots, m\}$ ,  $A_i$  and  $B_i$  for  $i \in \mathcal{M}$  are matrices with compatible dimensions, and  $U(t)$  is the control process. One can observe that the system matrices are subject to a random switching influence. At any given instance, these coefficient matrices are chosen from a set  $\mathcal{M}$  with a finite

number of candidates. The selection rule is dictated by the modulating switching process  $\alpha(t)$  that jump changes from one state to another at random times. Such systems have enjoyed numerous applications in emerging application areas as financial engineering, wireless communications, as well as in existing applications. A particular important problem concerns the asymptotic behavior of such systems when they are in operations for a long time. Our interest lies in finding admissible controls so that the resulting system will be almost surely stabilized. An added difficulty is that the process  $X(t)$  can only be observed with an additive noise

$$dX(t) = [A_{\alpha(t)}X(t) + B_{\alpha(t)}U(t)]dt + dW(t).$$

For such partially observed systems, it is natural to use nonlinear filtering techniques. The associated filter is known as the Wonham filter [25], which is one of a handful of finite dimensional filters in existence. Note that under suitable conditions, e.g., when the feedback control as a function of the state  $x$  grows at most linearly in  $x$ , the above stochastic differential equation with Markov switching has a unique solution almost surely. The existence and uniqueness of Markov switching diffusion is deemed to be well known, see for example, [21, Theorem 3.15]. Earlier work on stochastic differential equations with switching may be found in [24].

Having appealing and simple structures, linear quadratic regulators have had numerous applications. The applications on the other hand have stimulated further developments in control and systems theory. As a result, there is a vast literature on linear systems theory containing both time-honored results and very recent progress on a wide range of related topics. For example, some recent progress on continuous-time adaptive controls and output-feedback stabilization can be found in [6], [17] and references therein.

This paper is concerned with adaptive control systems with a hidden Markov chain, which can be viewed as a stochastic hybrid system. Our investigation is motivated by the previous work on stabilization of such systems [2], [7], see also [1], [9], [22] for some related control, estimation, and optimization problems for hybrid systems. Emerging applications have also been found in manufacturing systems [23], in which a Markov chain is used to represent the capacity of an unreliable machine, in wireless communication, where a Markov chain is used to depict randomly time varying signals or channels. Applications have also been found in queueing systems, biological systems, and financial and economic systems. For instance, in financial engineering, a geometric Brownian motion model

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for a stock is frequently used. The well-known Markowitz's mean-variance portfolio selection is one of the linear stochastic control problems. The traditional setup can be described by a linear stochastic differential equation, where both the appreciation rate and volatility are constants. Nevertheless, it has been recognized that such a formulation is far from realistic. Very often, there are additional randomness due to the variation of interest rates, volatility, and other random environment factors. For the mean-variance problem mentioned, some recent effort has been on obtaining optimal portfolio selections when both the appreciation rate and volatility depend on a Markov chain [27]. For all of the applications mentioned above, practical considerations often lead to the inclusion of unobservable Markov chains. For some recent study of hidden Markov chains, we refer the reader to [8] and references therein. Since a Markov chain is frequently used to conveniently model random environment changes and other factors that are not representable as the usual continuous dynamics, treating adaptive controls, stability, and stabilization of adaptive systems with a hidden Markov chain have significant impact on various applications.

There have been continued interest in dealing with hybrid systems under a Markov switching. In [22], stabilization for robust controls of jump linear systems was investigated. In [9], both controllability and stabilizability of jump linear systems were considered. In a recent paper [4], based on the vanishing discount approach to discounted functional of filtering process, some ergodicity results were obtained. Stability under random perturbations of Markov chain type can be traced back to the work [11]. This line of work has been substantially expanded to diffusion systems in [12], [14]. Recently, renewed interests have been shown to deal with switching diffusions; see for example [5], [13], [19], [20], [28] among others. These results include stability as well as ergodicity of general switching diffusion processes.

In many applications, one is mainly interested in if the systems under consideration are stable. A widely recognized, important task is to find control strategies so that the resulting systems are stable in an appropriate sense. In the literature, stabilization of continuous-time, adaptive control systems with hidden Markov chains were considered in [2], [7]. In both of these references, averaging criteria were used for the purpose of stabilization. To be more precise, adaptive control strategies were developed in [7] to make both the system and the control have bounded second moment in the sense

$$\limsup_{t \rightarrow \infty} \mathbb{E}[|X(t)|^2 + |U(t)|^2] < \infty,$$

whereas adaptive controls were obtained in [2] to have the second moments of the averages of both the system and control bounded in the sense

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t (|X(s)|^2 + |U(s)|^2) ds \right] < \infty.$$

Using these criteria enables to stabilize the system in certain expectation sense. A closer scrutiny of the system behavior necessarily requires the consideration of sample path properties.

In comparison with the aforementioned references, it is a worthwhile effort to examine the pathwise stabilization under partial observations. Our aim is the almost sure stability of the system in the sense of Definition 4.1. To this end, a log quadratic Lyapunov function is introduced in equation (4.9) to show that the feedback control law defined by (4.12) is stabilizing. Just as in elementary probability theory, we need to deal with different mode of convergence. Here, we focus on the almost sure stability. Note that under averaging criteria, a system is stabilizable but it may not be stabilizable for its sample paths. To be of any practical use in applications, the system resulting from an adaptive control law should not allow wild behavior in the sample paths. That is, the system should not have finite explosion time, which is known as regularity. Thus, it is imperative to find sufficient conditions for non-explosiveness or regularity. Owing to the use of adaptive control strategies, known results in stability and stabilization in Markov-modulated stochastic systems cannot be applied directly. As a result, pathwise stabilization is more difficult to deal with compared with that under averaging criteria. As will be seen in later section, the feedback adaptive controls render difficulty in analyzing the underlying systems. Certain functions associated with the diffusion matrix in fact grow faster than normally is allowed in the standard analysis. When averaged criteria are used, this kind of difficulty will not show up since by taking expectation, we can easily average out the Brownian motion term. However, when pathwise criteria are used, we can no longer use the argument based on using expectations. Thus the consideration of pathwise stabilization is both practically necessary and theoretically interesting.

For a practical system, no finite explosion time is a must. Following the convention in the theory of stochastic processes, a system without finite explosion time with probability one is termed regular in this paper. Next, for a deterministic system given by a differential equation, if the solutions are ultimately uniformly bounded, then it is Lagrange stable. For stochastic systems, almost sure boundedness excludes many useful cases (for example, any systems perturbed by a white noise). Thus, in lieu of such a boundedness, one seeks stability in certain weak sense. Starting from a point outside of a compact set, the system should be able to return to the set infinitely often with probability one. Such a property is known as recurrence. Note that the return time is random. For practical control systems, the average (or expected) return time to the compact set cannot be infinitely long otherwise the controlled system is useless. We say that the process is positive recurrent if the expected return time is finite. In fact, positive recurrence was termed weak stability for diffusion processes in [26]. Both regularity and recurrence of adaptive control systems can be viewed as "practical" stability conditions. To begin our quest of finding admissible controls that stabilize the systems almost surely, we examine if the controlled process is regular. We establish regularity under feedback controls with linear growth conditions. Then, we develop sufficient conditions and admissible adaptive controls under which the system is stabilizable. Moreover, as a by-product, we also establish positive recurrence of the underlying processes as a corollary of our stabilization result.

The rest of the paper is organized as follows. Section 2 presents the problem setup while Section 3 is devoted to the regularity of the underlying process. We show that, as long as the feedback controls have linear growth, the resulting systems is regular. Section 4 investigates the almost sure stability of the process. Some numerical simulations are also provided. We conclude the paper in Section 5 by some additional remarks. All technical proofs are postponed to the appendices.

## 2. FORMULATION AND PRELIMINARY

### A. Problem Setup

Denote by  $(\Omega, \mathcal{A}, \mathbb{P})$  a probability space with an associated nondecreasing family of  $\sigma$ -algebras  $(\mathcal{F}_t)$ . Let  $\alpha(t)$  be a continuous-time Markov chain with a finite state space  $\mathcal{M} = \{1, \dots, m\}$  and transition rate matrix  $\Pi = (\pi_{ij}) \in \mathbb{R}^{m \times m}$ , and  $W(t)$  be a standard  $\mathbb{R}^n$ -valued Brownian motion. In the above and hereafter,  $A'$  denotes the transpose of a matrix  $A$ ,  $|A| = \sqrt{\text{tr}(AA')}$  is the trace norm of  $A$ , and  $|v| = \sqrt{v'v}$  is the usual Euclidean norm of a vector  $v$ .

Assume throughout the paper that  $W(t)$  and  $\alpha(t)$  are independent. Let  $X(t) \in \mathbb{R}^n$  and  $U(t) \in \mathbb{R}^d$  be the state and control processes, respectively. For  $i \in \mathcal{M}$ ,  $A_i \in \mathbb{R}^{n \times n}$  and  $B_i \in \mathbb{R}^{n \times d}$  are matrices with appropriate dimensions. Our main interest focuses on the following regime-switching stochastic system

$$dX(t) = [A_{\alpha(t)}X(t) + B_{\alpha(t)}U(t)]dt + dW(t) \quad (2.1)$$

with square integrable initial condition  $X(0) = x$ . As in [2], [7], denoting the column vector of  $\mathbb{R}^m$  of indicator functions by

$$\Phi(t) = (\mathbb{I}_{\{\alpha(t)=1\}}, \dots, \mathbb{I}_{\{\alpha(t)=m\}})'$$

where  $\mathbb{I}_E$  stands for the usual indicator function of the event  $E$ , we may present the dynamics of the Markov chain by

$$d\Phi(t) = \Pi' \Phi(t)dt + dM(t).$$

The process  $M(t)$  is an  $\mathbb{R}^m$ -valued square integrable martingale with right continuous trajectories. The independence of  $\alpha(t)$  and  $W(t)$  implies that of  $\Phi(t)$  and  $W(t)$ . In all the sequel, we also assume that  $x$ ,  $\Phi(t)$ , and  $W(t)$  are mutually independent.

One of the main features of the system considered here is that the Markov chain under consideration is a hidden one. As treated in [2], [7], the essence is that we are dealing with a system (2.1) with unknown mode that switches back and forth among a finite set at random times. But different from previous consideration, we wish to establish the regularity of the process and to find conditions ensuring almost sure stabilization. The almost sure stabilization poses new challenges and difficulties since we cannot average out the martingale term by means of taking expectations. Compared with the aforementioned papers, different techniques are needed. Here, the keystone is to find a suitable Lyapunov function.

*Remark 2.1:* Throughout the paper, the process  $X(t)$  is assumed to be observable, but this is not the case for the switching process  $\alpha(t)$ . The problem belongs to the category of controls with partial observations. Observing  $\alpha(t)$  through

the adaptive control process with Gaussian white noise brings us to the framework of Wonham filtering problems [25].

*Definition 2.2:* A feedback control  $U$  is said to be admissible if for any  $t \geq 0$ ,  $U(t)$  is  $\mathcal{F}_t^X$ -measurable where the  $\sigma$ -algebra  $\mathcal{F}_t^X$  is given by  $\mathcal{F}_t^X = \sigma\{X(s), s \leq t\}$ .

We are now in position to state precisely the problem we wish to investigate.

Our purpose is to study the following problem.

- 1) We analyze (2.1) and obtain conditions under which the system will be regular. Hence, our goal is to propose sufficient conditions ensuring the process will not have finite explosion time. We show that, as long as the feedback control  $U$  grows at most linearly as a function of  $X$ , the resulting adaptive control system will be regular.
- 2) We design admissible adaptive controls and provide sufficient conditions that stabilize the closed-loop system almost surely. Loosely, the sufficient condition ensures that for almost all sample points, the corresponding system will be stabilizable. The precise definition of almost sure stabilization will be provided in the next section.

### B. Preliminary

As in [2], [7], we convert this partially observed system to a control process with complete observation. It entails to replace the hidden state  $\Phi(t)$  by its estimator, namely the well-known Wonham filter  $\hat{\Phi}(t)$ . Using feedback control  $U(t) = U(X(t), \hat{\Phi}(t))$ , we shall need the following notation

$$\begin{aligned} \hat{\Phi}_i(t) &= \mathbb{E}[\mathbb{I}_{\{\alpha(t)=i\}} | \mathcal{F}_t^X], \\ \hat{\Phi}(t) &= (\hat{\Phi}_1(t), \dots, \hat{\Phi}_m(t))' \in \mathbb{R}^m, \\ C(X(t)) &= (A_1X(t) + B_1U(t), \dots, A_mX(t) + B_mU(t)), \\ D(\varphi) &= (\text{diag}(\varphi) - \varphi\varphi') \text{ for } \varphi \in \mathbb{R}^m, \\ \text{diag}(\varphi) &= \text{diag}(\varphi_1, \dots, \varphi_m). \end{aligned} \quad (2.2)$$

Denote also the innovation process by

$$dV(t) = dX(t) - C(X(t))\hat{\Phi}(t)dt.$$

Using nonlinear filtering techniques [16], we can convert the partially observed control problem to a completely observable system

$$\begin{pmatrix} dX(t) \\ d\hat{\Phi}(t) \end{pmatrix} = \begin{pmatrix} C(X(t))\hat{\Phi}(t) \\ \Pi'\hat{\Phi}(t) \end{pmatrix} dt + \begin{pmatrix} I_n \\ D(\hat{\Phi}(t))C(X(t))' \end{pmatrix} dV(t), \quad (2.3)$$

where  $I_n$  stands for the identity matrix of order  $n$ . Note that the above representation is easier to work with in the subsequent analysis. To solve the equations, we begin with the Wonham filtering equation. After  $\hat{\Phi}(t)$  being found, we use it in the first equation. Under suitable conditions, e.g., linear feedback controls, the resulting stochastic differential equation will have a unique solution in the almost sure sense.

*Remark 2.3:* Before proceeding further, we shall make a few remarks. The form  $C(X(t))$  indicates the  $X(t)$ -dependence. When the feedback control  $U(t)$  is of linear form, it depends on  $X(t)$  linearly. This point will be used in what follows. The equivalent and completely observable system can be viewed as a controlled diffusion, in which the usual diffusion term is replaced by

$$\begin{pmatrix} I_n \\ D(\widehat{\Phi}(t))C(X(t))' \end{pmatrix}$$

and the driven Brownian motion is given by  $V(t)$ . When linear feedback control is used, both the drift and diffusion grow at most linearly, which is a useful observation. Since  $\widehat{\Phi}(t)$  is the probability conditioned on the observation, for each  $t \geq 0$  and each  $i \in \mathcal{M}$ ,  $\widehat{\Phi}_i(t) \geq 0$  with

$$\sum_{i=1}^m \widehat{\Phi}_i(t) = 1.$$

Denote the joint vector by  $Y(t) = (X(t), \widehat{\Phi}(t))' \in \mathbb{R}^{n+m}$ . In what follows, we often consider  $|Y(t)| \geq r$  for some  $r > 0$ , where  $|Y|$  is the usual Euclidean norm. Denote by  $N(0; r) \in \mathbb{R}^{n+m}$  the neighborhood centered at 0 with radius  $r$ . Using the notation defined in (2.2) associated with the stochastic differential equation (2.3), we define the following operator. For each sufficiently smooth real-valued function  $h : \mathbb{R}^{n+m}/N(0; r) \mapsto \mathbb{R}$ , let

$$\begin{aligned} \mathcal{L}h(y) &= \mathcal{L}h(x, \varphi) \\ &= (\nabla h(x, \varphi))' \begin{pmatrix} C(x)\varphi \\ \Pi'\varphi \end{pmatrix} \\ &\quad + \frac{1}{2} \text{tr} \left( (I_n \ C(x)D(\varphi)') \nabla^2 h(x, \varphi) (I_n \ C(x)D(\varphi)')' \right) \end{aligned} \quad (2.4)$$

where  $\nabla h$  and  $\nabla^2 h$  are the gradient and Hessian of  $h$ , respectively.

### 3. REGULARITY

First, let us recall the definition of regularity. According to [12], the Markov process

$$Y(t) = \begin{pmatrix} X(t) \\ \widehat{\Phi}(t) \end{pmatrix}$$

is regular, if for any  $0 < T < \infty$ ,

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} |Y(t)| = \infty \right) = 0.$$

Roughly speaking, regularity ensures the process under consideration will not have finite explosion time. For our adaptive control systems, we proceed to show that under linear feedback control, the systems is regular.

*Theorem 3.1:* Assume that the feedback control  $U(t) = U(X(t), \widehat{\Phi}(t))$  is admissible and that for all  $t \geq 0$ ,  $|U(t)| \leq K(1 + |X(t)|)$  for some  $K > 0$ . Then, the feedback control system (2.3) is regular.

*Remark 3.2:* The condition above indicates that  $U(x, \varphi)$  grows at most linearly in  $x$ . In fact, for our problem, we are mainly interested in linear feedback controls in the variable  $x$ . In this case, the linear growth condition is clearly satisfied.

**Proof.** Let  $\Theta$  be an open set in  $\mathbb{R}^{n+m}$  and denote  $O = \left\{ y = (x, \varphi)' \in \Theta, \varphi = (\varphi_1, \dots, \varphi_m) \text{ satisfying } \varphi_i \geq 0 \text{ for } i \in \mathcal{M}, \text{ and } \sum_{i=1}^m \varphi_i = 1 \right\}$ . We first observe that both the drift and the diffusion coefficient given in (2.3) satisfy the linear growth and Lipschitz condition in every open set in  $O \subset \mathbb{R}^{n+m}$ . Thus, to prove the regularity, using the result in [12], we only need to show that there is a nonnegative function  $\mathcal{U}$  which is twice continuously differentiable in  $O_r = \{y \in O, |y| > r\}$  for some  $r > 0$  with  $y = (x, \varphi)'$  such that

$$\inf_{|y| > R} \mathcal{U}(y) \rightarrow \infty \text{ as } R \rightarrow \infty, \quad (3.1)$$

and that there is an  $\gamma > 0$  satisfying

$$\mathcal{L}\mathcal{U}(y) \leq \gamma \mathcal{U}(y). \quad (3.2)$$

Thus, to verify the regularity of the process  $Y(t)$ , all needed is to construct an appropriate Lyapunov function  $\mathcal{U}$ . Note that we only need a Lyapunov function that is smooth and defined in the complement of a sphere. Equivalently, we only need the smoothness of the Lyapunov function to be in a deleted neighborhood of the origin. To this end, take  $r = 1$  and denote by  $O_1$  the set

$$\begin{aligned} O_1 &= \left\{ y = (x, \varphi)' \in \mathbb{R}^{n+m}, |y| > 1 \text{ and } \varphi = (\varphi_1, \dots, \varphi_m) \right. \\ &\quad \left. \text{satisfying } \varphi_i \geq 0 \text{ for } i \in \mathcal{M}, \text{ and } \sum_{i=1}^m \varphi_i = 1 \right\}. \end{aligned} \quad (3.3)$$

Define  $\mathcal{U} : O_1 \mapsto \mathbb{R}$  as  $\mathcal{U}(y) = |y|$ . It is easily checked that condition (3.1) holds. Moreover, we have

$$\nabla \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right| = \frac{\begin{pmatrix} x \\ \varphi \end{pmatrix}}{\left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|},$$

and

$$\nabla^2 \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right| = \frac{I_{n+m}}{\left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|} - \frac{\begin{pmatrix} xx' & x\varphi' \\ \varphi x' & \varphi\varphi' \end{pmatrix}}{\left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|^3}.$$

Consequently, it follows from (2.4) that

$$\begin{aligned} \mathcal{L}\mathcal{U}(x, \varphi) &= \frac{1}{\left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|} (x'C(x)\varphi + \varphi'\Pi'\varphi) \\ &\quad + \frac{1}{2 \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|} (n + \text{tr}(D(\varphi)C'(x)C(x)D'(\varphi))) \\ &\quad - \frac{1}{2 \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|^3} \text{tr} \left( (I_n \ C(x)D(\varphi)') \begin{pmatrix} xx' & x\varphi' \\ \varphi x' & \varphi\varphi' \end{pmatrix} \right. \\ &\quad \left. \times (I_n \ C(x)D(\varphi)')' \right). \end{aligned} \quad (3.4)$$

Note that the set that we are working with is  $O_1$  defined in (3.3). In particular, the use of  $O_1$  yields that for any  $y \in O_1$ ,  $|\varphi|$  is always bounded. We also note that owing to the definition of  $C(x)$  and the linear growth feedback controls used,  $C(x)$  is a function grows at mostly linearly in  $x$ . It

follows that for the terms on the third line from bottom of (3.4), for  $|(x, \varphi)'|$  large enough, there is a constant  $\gamma_1 > 0$ ,

$$\begin{aligned} & \frac{1}{\left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|} \left| x' C(x) \varphi + \varphi' \Pi' \varphi \right| \\ & \leq \frac{\gamma_1}{\left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|} \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|^2 \leq \gamma_1 \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|. \end{aligned}$$

Likewise, for the next two terms, we have for some  $\gamma_2 > 0$ ,

$$\frac{1}{2 \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|} \left| n + \text{tr}(D(\varphi) C'(x) C(x) D'(\varphi)) \right| \leq \gamma_2 \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right|.$$

Combining the above estimates, we can deduce that for some  $\gamma > 0$ ,

$$\mathcal{L}U(x, \varphi) \leq \gamma \left| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right| = \gamma \mathcal{U}(x, \varphi).$$

Consequently, the second condition (3.2) is satisfied and the regularity of the feedback control is established.  $\square$

#### 4. STABILIZATION

In this section, we establish conditions under which the system of interest is stabilizable in the almost sure sense. We first present the definition and then proceed to find sufficient conditions for stabilization.

*Definition 4.1:* System (2.1) or equivalently (2.3) is said to be almost surely stabilizable if there is a feedback control law  $U(t)$  such that the resulting trajectories satisfy

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t)| \leq 0 \quad \text{almost surely.} \quad (4.1)$$

Note that the definition given in (4.1) is natural. When studying stability of stochastic differential equations, especially for pathwise stability, one uses the so-called  $q$ th-moment Lyapunov exponent

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t)|^q$$

for some  $q > 0$ . Here, roughly, we require that under the control law, the first-moment Lyapunov exponent is non-positive.

##### A. Auxiliary Results

Before proceeding further, let us first recall a lemma, which is concerned with the existence of the associated system of Riccati equations when quadratic cost criteria are used. The proof of the lemma is given in [10].

*Lemma 4.2:* Consider the system of Riccati equations

$$A_i' P_i + P_i A_i - P_i B_i R^{-1} B_i' P_i + \sum_{j=1}^m \pi_{ij} P_j + Q = 0, \quad i \in \mathcal{M}, \quad (4.2)$$

where  $Q \in \mathbb{R}^{n \times n}$  is symmetric and positive semi-definite, and  $R \in \mathbb{R}^{m \times m}$  is symmetric and positive definite. The system

(4.2) has a solution if and only if for each  $i \in \mathcal{M}$ , there is a matrix  $\bar{P}_i$  satisfying

$$A_i' \bar{P}_i + \bar{P}_i A_i - \bar{P}_i B_i R^{-1} B_i' \bar{P}_i + \sum_{j=1}^m \pi_{ij} \bar{P}_j + Q \leq 0. \quad (4.3)$$

Furthermore, if  $Q$  is positive definite, so are  $P_i$  for  $i \in \mathcal{M}$ .

To carry out the analysis, we need some auxiliary results on the bounds of the quadratic variation process. Before getting the almost sure bounds, we examine the moment bounds for certain related martingales, which turn out to be interesting in their own right. The main ingredient is the use of properties of the associated Markov chain.

##### Moment Bounds

*Proposition 4.3:* Consider the stochastic differential equation

$$d\hat{\Phi}(t) = \Pi' \hat{\Phi}(t) dt + D(\hat{\Phi}(t)) C(X(t))' dV(t) \quad (4.4)$$

and define the associate martingale

$$N(t) = \int_0^t D(\hat{\Phi}(s)) C(X(s))' dV(s). \quad (4.5)$$

Suppose that the Markov chain  $\alpha(t)$  is irreducible. Then, for some positive constant  $K$  independent of  $t$ ,

$$\mathbb{E} \left[ \frac{1}{t} |N(t)|^2 \right] \leq K. \quad (4.6)$$

**Proof.** The proof is given in Appendix A.  $\square$

*Remark 4.4:* It follows from the proof of Proposition 4.3 that the limit of the matrix

$$S = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int_0^t \Pi' [\hat{\Phi}(u) - \nu] [\hat{\Phi}(s) - \nu]' \Pi ds du$$

is finite. Clearly, this matrix is symmetric and positive semi-definite. A moment of reflect reveals that we can further prove the asymptotic normality in that as  $t \rightarrow \infty$ ,

$$\frac{1}{\sqrt{t}} \int_0^t \Pi' [\hat{\Phi}(s) - \nu] ds \text{ converges in distribution to } \mathcal{N}(0, S).$$

That is, a normalized sequence defined on the left-hand side above converges in distribution to a normal random vector with mean 0 and covariance  $S$ .

Another ramification is that in lieu of considering the second-moment bounds, we can deal with  $q$ th-moment bounds. In fact, using the same techniques, we can show that for any integer  $p > 0$ ,

$$\mathbb{E} \left[ \left| \frac{1}{\sqrt{t}} \int_0^t \Pi' [\hat{\Phi}(s) - \nu] ds \right|^{2p} \right] < \infty.$$

Hence, as the solution of (4.4) is given by

$$\hat{\Phi}(t) = \hat{\Phi}(0) + \int_0^t \Pi' \hat{\Phi}(s) ds + N(t)$$

which means that

$$N(t) = \hat{\Phi}(t) - \hat{\Phi}(0) - \int_0^t \Pi' \hat{\Phi}(s) ds,$$

we obtain that

$$\mathbb{E} \left[ \left| \frac{1}{\sqrt{t}} N(t) \right|^{2p} \right] < \infty.$$

Next, for odd exponents and for any integer  $p \geq 1$ , it follows from Hölder's inequality that

$$\left( \mathbb{E} \left[ \left| \frac{1}{\sqrt{t}} N(t) \right|^{2p-1} \right] \right)^{2p} \leq \left( \mathbb{E} \left[ \left| \frac{1}{\sqrt{t}} N(t) \right|^{2p} \right] \right)^{2p-1} < \infty.$$

Finally, we conclude that for any positive integer  $q$ ,

$$\mathbb{E} \left[ \left| \frac{1}{\sqrt{t}} N(t) \right|^q \right] < \infty.$$

### Almost Sure Bounds

For the almost sure stabilization, we need to show that the quadratic variation of  $N(t)$  satisfies

$$\frac{1}{t} \langle N, N \rangle_t \leq K \text{ a.s.}$$

for some  $K > 0$  independent of  $t$ .

*Proposition 4.5:* Consider (4.4) and suppose that the Markov chain  $\alpha(t)$  is irreducible. Then, the quadratic variation of the process  $N(t)$  satisfies  $\langle N, N \rangle_t \leq Kt$  where  $K$  is some positive constant independent of  $t$ . Therefore,

$$\lim_{t \rightarrow \infty} \frac{1}{t} N(t) = 0 \text{ a.s.} \quad (4.7)$$

**Proof.** The proof is given in Appendix B.  $\square$

### B. Stabilization

*Lemma 4.6:* Consider the set  $\Delta$  defined by

$$\Delta = \left\{ (x, \varphi) \in \mathbb{R}^n \times \mathbb{R}^m, \varphi = (\varphi_1, \dots, \varphi_m) \right. \\ \left. \text{satisfying } \varphi_i \geq 0 \text{ and } \sum_{i=1}^m \varphi_i = 1 \right\}.$$

Denote

$$P(\varphi) = \sum_{i=1}^m P_i \varphi_i. \quad (4.8)$$

For some  $\theta > 0$ , let  $V_\theta(x, \varphi) : \Delta \mapsto \mathbb{R}$  with

$$V_\theta(x, \varphi) = \log(\theta + x' P(\varphi) x). \quad (4.9)$$

Then, we have

$$\mathcal{L}V_\theta(x, \varphi) = \frac{1}{\theta + x' P(\varphi) x} (2x' P(\varphi) C(x) \varphi + (x' \tilde{P} x)' \Pi' \varphi) \\ + \frac{1}{\theta + x' P(\varphi) x} \text{tr} \left( P(\varphi) + 2C(x) D(\varphi)' x' \tilde{P} \right) \\ - \frac{1}{2(\theta + x' P(\varphi) x)^2} \\ \times \text{tr} \left( (I_n, C(x) D(\varphi)') \Lambda(x, \varphi) (I_n, C(x) D(\varphi)')' \right), \quad (4.10)$$

where

$$\Lambda(x, \varphi) = \begin{pmatrix} 2P(\varphi)x \\ x' \tilde{P} x \end{pmatrix} \begin{pmatrix} 2P(\varphi)x \\ x' \tilde{P} x \end{pmatrix}', \\ P = (P_1, \dots, P_m)', \quad x' \tilde{P} x = (x' P_1 x, \dots, x' P_m x)' \in \mathbb{R}^m, \\ P x = (P_1 x, \dots, P_m x) \in \mathbb{R}^{n \times m}, \quad x' \tilde{P} = (\tilde{P} x)' \in \mathbb{R}^{m \times n}.$$

**Proof.** We have

$$\nabla \log(\theta + x' P(\varphi) x) = \frac{\begin{pmatrix} 2P(\varphi)x \\ x' \tilde{P} x \end{pmatrix}}{\theta + x' P(\varphi) x}$$

and

$$\nabla^2 \log(\theta + x' P(\varphi) x) \\ = - \frac{\begin{pmatrix} 2P(\varphi)x \\ x' \tilde{P} x \end{pmatrix} \begin{pmatrix} 2P(\varphi)x \\ x' \tilde{P} x \end{pmatrix}'}{(\theta + x' P(\varphi) x)^2} + \frac{2 \begin{pmatrix} P(\varphi) & \tilde{P} x \\ x' \tilde{P} & 0_m \end{pmatrix}}{\theta + x' P(\varphi) x},$$

where  $0_m$  stands for a square matrix of order  $m$  with all entries equal to zero. Consequently, it follows from (2.4) that

$$\mathcal{L}V_\theta(x, \varphi) = \frac{1}{\theta + x' P(\varphi) x} \left( 2x' P(\varphi) C(x) \varphi + (x' \tilde{P} x)' \Pi' \varphi \right) \\ + \frac{1}{2} \text{tr} \left( (I_n, C(x) D(\varphi)') \nabla^2 V_\theta(x, \varphi) (I_n, C(x) D(\varphi)')' \right), \quad (4.11)$$

which immediately implies (4.10).  $\square$

For the purpose of stabilization, we also need an estimate on  $\mathcal{L}V_\theta(X(t), \hat{\Phi}(t))$ .

*Lemma 4.7:* Assume that equation (4.3) is satisfied and that

$$Q - \frac{1}{2} [P_i B_i - P_j B_j] R^{-1} [P_i B_i - P_j B_j]'$$

are positive definite matrices for all  $(i, j) \in \mathcal{M}^2$  where  $P_i$  for  $i \in \mathcal{M}$  are the solutions of the algebraic Riccati equations given by (4.2). Then, the infinitesimal generator of the process  $(X(t), \hat{\Phi}(t))$  associated with the feedback control law

$$U(t) = -R^{-1} \sum_{i=1}^m \hat{\Phi}_i(t) B_i' P_i X(t), \quad (4.12)$$

satisfies for some constant  $\gamma > 0$ ,

$$\mathcal{L}V_\theta(X(t), \hat{\Phi}(t)) \leq \frac{\gamma}{\theta}. \quad (4.13)$$

**Proof.** We can deduce from Lemma 4.6 that

$$\mathcal{L}V_\theta(X(t), \hat{\Phi}(t)) \\ \leq \frac{1}{\theta + X(t)' P(\hat{\Phi}(t)) X(t)} (2X(t)' P(\hat{\Phi}(t)) C(X(t)) \hat{\Phi}(t)) \\ + \frac{1}{\theta + X(t)' P(\hat{\Phi}(t)) X(t)} ((X(t)' \tilde{P} X(t))' \Pi' \hat{\Phi}(t)) \\ + \frac{1}{\theta + X(t)' P(\hat{\Phi}(t)) X(t)} \\ \text{tr} (P(\hat{\Phi}(t)) + 2C(X(t)) D(\hat{\Phi}(t))' X(t)' \tilde{P}).$$

Therefore, following exactly the same lines as in [7], we obtain

$$\mathcal{L}V_\theta(X(t), \hat{\Phi}(t)) \leq - \frac{1}{\theta + X(t)' P(\hat{\Phi}(t)) X(t)} \left( X(t)' \left[ Q \right. \right. \\ \left. \left. - \sum_{i=1}^m \sum_{j=1}^m \frac{\hat{\Phi}_i(t) \hat{\Phi}_j(t)}{2} [P_i B_i - P_j B_j] R^{-1} [P_i B_i - P_j B_j]' \right. \right. \\ \left. \left. + \left( \sum_{j=1}^m \hat{\Phi}_j(t) B_j' P_j \right)' R^{-1} \left( \sum_{i=1}^m \hat{\Phi}_i(t) B_i' P_i \right) \right] X(t) - \text{tr}(P(\hat{\Phi}(t))) \right).$$

Finally,

$$\mathcal{L}V_\theta(X(t), \hat{\Phi}(t)) \leq \frac{1}{\theta} \sum_{i=1}^m \text{tr}(P_i)$$

which completes the proof of Lemma 4.7.  $\square$

*Theorem 4.8:* Assume that the conditions of Lemma 4.7 are satisfied. Then, the feedback control law defined in equation (4.12) stabilizes system (2.3) almost surely.

**Proof.** It follows from Ito's rule that

$$V_\theta(X(t), \widehat{\Phi}(t)) = V_\theta(x, \varphi) + \int_0^t \mathcal{L}V_\theta(X(s), \widehat{\Phi}(s))ds + M(t) \quad (4.14)$$

with the initial condition  $X(0) = x$  and  $\widehat{\Phi}(0) = \varphi$  and the martingale term

$$M(t) = \int_0^t \Sigma(s)dV(s)$$

where

$$\begin{aligned} \Sigma(s) &= \frac{1}{\Theta + X(s)'P(\widehat{\Phi}(s))X(s)} \\ &\times (2X(s)'P(\widehat{\Phi}(s)), (X(s)'\tilde{P}X(s))') \begin{pmatrix} I_n \\ D(\widehat{\Phi}(s))C(X(s))' \end{pmatrix}, \text{ which leads to} \\ &= \frac{1}{\theta + X(s)'P(\widehat{\Phi}(s))X(s)} \\ &\times (2X(s)'P(\widehat{\Phi}(s)) + (X(s)'\tilde{P}X(s))'D(\widehat{\Phi}(s))C(X(s))'). \end{aligned}$$

We can split the martingale  $M(t)$  into two terms,  $M(t) = N_1(t) + N_2(t)$  with

$$\begin{aligned} N_1(t) &= \int_0^t \frac{2X(s)'P(\widehat{\Phi}(s))}{\theta + X(s)'P(\widehat{\Phi}(s))X(s)} dV(s), \\ N_2(t) &= \int_0^t \frac{(X(s)'\tilde{P}X(s))'}{\theta + X(s)'P(\widehat{\Phi}(s))X(s)} \\ &\quad \times D(\widehat{\Phi}(s))C(X(s))' dV(s). \end{aligned}$$

It is easy to see that

$$\frac{4X(t)'P(\widehat{\Phi}(t))P(\widehat{\Phi}(t))X(t)}{(\theta + X(t)'P(\widehat{\Phi}(t))X(t))^2} \leq K_1$$

where  $K_1 = \frac{m}{\theta} \max_{i \in \mathcal{M}} (\lambda_{\max}(P_i))$ . Then, the quadratic variation of  $N_1(t)$  satisfies  $\langle N_1, N_1 \rangle_t \leq K_1 t$  a.s. Consequently, we deduce from the strong law of large numbers for local martingales [15] that

$$\lim_{t \rightarrow \infty} \frac{1}{t} N_1(t) = 0 \quad \text{a.s.} \quad (4.15)$$

In view of Proposition 4.5, one can also find a positive constant  $K_2$ , independent of  $t$ , such that

$$\begin{aligned} \langle N_2, N_2 \rangle_t &= \int_0^t \frac{|X(s)'\tilde{P}X(s)|^2}{(\theta + X(s)'P(\widehat{\Phi}(s))X(s))^2} \\ &\quad \times |D(\widehat{\Phi}(s))C(X(s))'|^2 ds \leq K_2 t \quad \text{a.s.} \end{aligned} \quad (4.16)$$

It also ensures that

$$\lim_{t \rightarrow \infty} \frac{1}{t} N_2(t) = 0 \quad \text{a.s.} \quad (4.17)$$

Therefore, (4.15) and (4.17) imply that

$$\lim_{t \rightarrow \infty} \frac{1}{t} M(t) = 0 \quad \text{a.s.} \quad (4.18)$$

Thus, we find from (4.14) that

$$\frac{1}{t} V_\theta(X(t), \widehat{\Phi}(t)) = \frac{1}{t} V_\theta(x, \varphi) + \frac{1}{t} \int_0^t \mathcal{L}V_\theta(X(s), \widehat{\Phi}(s))ds + o(1) \quad \text{a.s.}$$

Moreover,  $V_\theta(x, \varphi)/t = o(1)$  as  $t \rightarrow \infty$  a.s. By virtue of Lemma 4.7, it follows that for all  $\theta > 0$

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} V_\theta(X(t), \widehat{\Phi}(t)) \\ = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathcal{L}V_\theta(X(s), \widehat{\Phi}(s))ds \leq \frac{\gamma}{\theta} \quad \text{a.s.} \end{aligned} \quad (4.19)$$

Furthermore, one can observe that  $x'P(\varphi)x \geq \lambda_{\min}(P(\varphi))|x|^2$  and since  $P(\varphi)$  is positive definite, the minimal eigenvalue of  $P(\varphi)$  is positive. Consequently,

$$\begin{aligned} \log(\lambda_{\min}(P(\varphi))) + 2\log(|x|) &\leq \log(\theta + \lambda_{\min}(P(\varphi))|x|^2) \\ &\leq \log(\theta + x'P(\varphi)x) \end{aligned}$$

$$\frac{1}{t} \left( \log(\lambda_{\min}(P(\widehat{\Phi}(t)))) + 2\log(|X(t)|) \right) \leq \frac{1}{t} V_\theta(X(t), \widehat{\Phi}(t)). \quad (4.20)$$

Finally, we conclude from (4.19) and (4.20) that for all  $\theta > 0$ ,

$$\limsup \frac{1}{t} \log |X(t)| \leq \frac{\gamma}{2\theta} \quad \text{a.s.}$$

We complete the proof of Theorem 4.8 by taking the limit as  $\theta$  tends to infinity.  $\square$

*Remark 4.9:* Normally, dealing with stochastic differential equations, to obtain the almost sure bounds of the solutions, one often relies on the use of appropriate Lyapunov functions to have the diffusion term of the process be bounded after a transformation. Here, we are dealing with a martingale term with some what faster rate of growth in  $x$ . Nevertheless, thanks to the second component of the diffusion (4.4), the probabilistic meaning of  $\widehat{\Phi}(t)$  enables us to work around the obstacle. To obtain the desired bounds, an alternative is to obtain an almost sure central limit theorem. Here however, we take a different approach. The main point is the use of Proposition 4.5.

Recall the notion of recurrence for the diffusion process  $(X(t), \widehat{\Phi}(t))$  starting at  $X(0) = x$  and  $\widehat{\Phi}(0) = \varphi$ . Consider an open set  $O$  with compact closure, and let

$$\sigma_O^{x, \varphi} = \inf \left\{ t > 0, (X(t), \widehat{\Phi}(t)) \in O \right\}$$

be the first entrance time of the diffusion to the set  $O$ , where  $\sigma_O^{x, \varphi}$  signifies the dependence on the initial data  $(x, \varphi)$  as well as on the set  $O$ . If  $(X(t), \widehat{\Phi}(t))$  is regular, it is recurrent with respect to  $O$  if  $\mathbb{P}\{\sigma_O^{x, \varphi} < \infty\} = 1$  for any  $(x, \varphi) \in O^c$ , where  $O^c$  is the complement of  $O$ . A recurrent process with finite mean recurrence time for some set  $O$ , is said to be positive recurrent with respect to  $O$ , otherwise, the process is null recurrent with respect to  $O$ . It has been proven in [12] that recurrence and positive recurrence are independent of the set  $O$  chosen. Thus, if it is recurrent (resp. positive recurrent) with respect to  $D$ , then it is recurrent (resp. positive recurrent) with respect to any other open set  $\Theta$  in the domain of interest.

Looking over the proof of the stabilization presented, we could show that for the Lyapunov function

$$V_0(x, \varphi) = \log(x' P(\varphi) x),$$

one can find  $\gamma > 0$  such that for all  $(x, \varphi) \in O^c$ ,

$$\mathcal{L}V_0(x, \varphi) \leq -\gamma. \quad (4.21)$$

In view of the known result of positive recurrence of diffusion processes [12], (4.21) is precisely a necessary and sufficient condition for positive recurrence. Thus, we obtain the following result as a by-product.

*Corollary 4.10:* Under the conditions of Theorem 4.8, with the control law (4.12) used, the diffusion systems (2.3) is positive recurrent.

We would like to add that the positive recurrence of the process is an important property. It has engineering implication for various applications. Essentially, it ensures that starting from a point outside of a bounded set, the control laws enables the system to return to a compact set almost surely. This may be viewed as a practical stability condition. In fact, Wonham used the term weak stability for such a property in his paper [26]. Dealing with diffusions, it was proved in [12] that positive recurrence leads to ergodicity. Treating switching diffusions with state-dependent switching, positive recurrence leading to ergodicity together with the characterization of the ergodic measures was provided in [28]. In our case, using the result of [12], positive recurrence also yields ergodicity.

### C. An Illustrative Example

Our goal is to illustrate Theorem 4.8 by numerical simulations. This example is taken from [7]. The state process takes values in  $\mathbb{R}^2$  and the control process is scalar which means that  $n = 2$  and  $d = 1$ . There are two modes with the generator

or transition rate matrix  $\Pi$  given by  $\Pi = \begin{pmatrix} -8 & 8 \\ 0.4 & -0.4 \end{pmatrix}$ .

In mode 1, the plant is governed by  $A_1 = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$ ,

$B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , while, in mode 2, by  $A_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ ,

$B_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ . The weighting matrices of the coupled

Riccati equations are chosen as  $Q = I_2$  and  $R = 10$ . One can check that the matrices  $\bar{P}_1 = 100I_2$  and  $\bar{P}_2 = 50I_2$  verify equation (4.3). The coupled Riccati equations then admit a unique solution which can be solved numerically by the algorithm proposed in [10]. We find that

$$P_1 = \begin{pmatrix} 0.4976 & 0.1868 \\ 0.1868 & 1.8361 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0.4974 & 0.2299 \\ 0.2299 & 0.8819 \end{pmatrix}$$

It is easy to verify that, as required in Lemma 4.7, the matrix

$$Q - \frac{1}{2}(P_1 B_1 - P_2 B_2)R^{-1}(P_1 B_1 - P_2 B_2)'$$

is definite positive. The following figures show that the feedback control  $U(t)$  defined by (4.12) can be applied to stabilize this system almost surely in the sense of equation (4.1). Figure 1 shows a sample path of  $\log |X(t)|/t$  where  $t$  runs over the interval  $(0, T)$  with  $T = 500$ , while Figure 2 is the zoom of this sample path. Our numerical results show that the feedback control  $U(t)$  performs well and that the system is almost surely stabilizable.

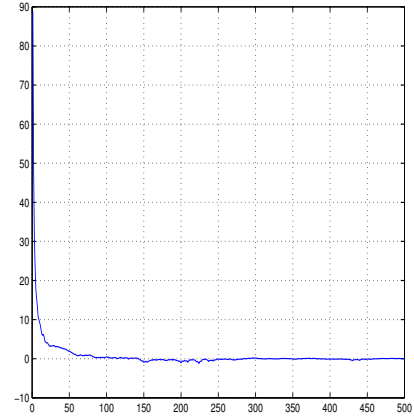


Fig. 1. Almost sure stability: A sample path of  $\log |X(t)|/t$  vs  $t$ ; horizontal axis represents  $t$  and vertical axis plots  $\log |X(t)|/t$

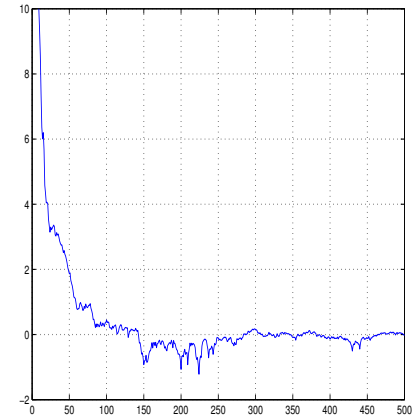


Fig. 2. Zoom of the almost sure stability: A sample path of  $\log |X(t)|/t$  vs  $t$ ; horizontal axis represents  $t$  and vertical axis plots  $\log |X(t)|/t$

## 5. FURTHER REMARKS

This paper has been concerned with stabilization in the almost sure sense of an adaptive control system with linear dynamics modulated by an unknown Markov chain. Under the framework of Wonham filtering, the underlying system is converted to a fully observable system. Using a feedback control that is linear in the continuous state variable, we establish pathwise stabilization of the process. Along the way of our study, we have also obtained regularity of the underlying process. In addition, as a corollary, we have shown that under



the stabilizing control law, the resulting system is positive recurrent. These results pave a way for practical consideration of stabilization of adaptive controls of linear systems with a hidden Markov chain. Several directions may be worthwhile for further study and investigation.

- In our study, irreducibility of the Markov chain is used. We note that the irreducibility ensures the spectrum gap condition or exponential decay in (A.6) and (A.9) of Proposition 4.3 to hold. It will be interesting to see if it is possible to remove this condition. Our initial thoughts are: Under certain conditions, this might be possible. For example, if the Markov chain has several irreducible classes such that the states in each class vary rapidly, and among different classes, they change slowly. One may be able to use the different time scales to overcome the difficulty under the framework of time-scale separation using a singular perturbation approach. However, the details on this need to be thoroughly worked out; they are in fact out of the scope of the current paper.
- A related problem of considerable interest is to develop optimal controls under partial observation for a long-run quadratic cost functional. So far there have been various attempts to deal with these and related issues partially observed stochastic control problems. To the best of our knowledge, the problem is largely open. Moreover, it is far from obvious that the proposed stabilizing control law defined in (4.12) would be optimal for a long-run quadratic cost functional. As normally is the case that stabilization and optimal controls are treated separately in the literature, the related optimal controls for the partially observed systems involving a hidden Markov chain require different techniques, careful thoughts, and further investigations.
- It will be interesting to design admissible controls and find sufficient conditions for stabilization of linear systems with a hidden Markov in discrete-time.
- In our setup, the process  $X(t)$  represents the noisy observation–hidden Markov chain observed in white noise. A class of controlled regime-switching diffusion systems provides a somewhat more complex setup. In such a system, the dynamics are represented by switching diffusions with a hidden Markov chain. The Markov chain is not observable but can only be observed in another Gaussian white noise. That is, let us consider the controlled system

$$\begin{aligned} dY(t) &= [A_{\alpha(t)}Y(t) + B_{\alpha(t)}U(t)]dt + \sigma_{\alpha(t)}dV(t) \\ dX(t) &= g_{\alpha(t)}dt + \rho(t)dW(t), \end{aligned} \quad (5.1)$$

where  $Y(t)$  and  $X(t)$  are vector-valued processes with compatible dimensions representing the state and observations, respectively,  $V(t)$  and  $W(t)$  are independent multi-dimensional Brownian motions, and  $\alpha(t)$  is the hidden Markov chain with a finite state space. As was alluded to in the introduction, one of the motivations is Markowitz's mean-variance portfolio selections [27]. One may then pose similar stabilization problems.

- Recently, using regime-switching jump diffusions, which are switching diffusions with additional external jumps of a compound Poisson process, for modeling surplus in insurance risk has drawn much attention. A related problem in the adaptive setup is a regime-switching jump diffusion system in which the hidden Markov chain is observed similar to the observation in (5.1). One may then proceed with the study of stabilization problems.
- In the study of stabilization, positive definiteness of certain matrices is used (see Lemma 4.2). A challenging problem is to investigate the stabilization problem with the positive definiteness removed for the system given by (5.1). Here, the crucial point seems to rely on recent developments in linear quadratic control problems with indefinite control weights [3]. One needs to use the backward stochastic differential equations from the toolbox of stochastic analysis.

All of these problems deserve careful study and investigation. They open up new domains for further investigations.

## APPENDIX A

This appendix is devoted to the proof of Proposition 4.3. It is divided into several steps.

**Step 1.** We already saw that the solution (4.4) is given by

$$\widehat{\Phi}(t) = \widehat{\Phi}(0) + \int_0^t \Pi' \widehat{\Phi}(s) ds + N(t).$$

Consequently

$$N(t) = \widehat{\Phi}(t) - \widehat{\Phi}(0) - \int_0^t \Pi' \widehat{\Phi}(s) ds. \quad (A.1)$$

In view of (A.1), the probabilistic interpretation of  $\widehat{\Phi}(t)$  implies that  $N(t)$  is a martingale bounded almost surely for each  $t > 0$ . We proceed to obtain the moment bounds of  $N(t)$ .

**Step 2.** As  $\Pi$  is the generator of the irreducible Markov chain  $\alpha(t)$ , its unique stationary distribution  $\nu$  satisfies  $\Pi' \nu = 0$ . Hence, it follows that

$$\int_0^t \Pi' \widehat{\Phi}(s) ds = \int_0^t \Pi' (\widehat{\Phi}(s) - \nu) ds.$$

From (4.5), we have

$$\mathbb{E}[|N(t)|^2] = \mathbb{E} \left[ \int_0^t |D(\widehat{\Phi}(s))C(X(s))'|^2 ds \right]. \quad (A.2)$$

On the other hand, we deduce from (A.1) that

$$\begin{aligned} \frac{1}{t} \mathbb{E}[|N(t)|^2] &= \frac{1}{t} \mathbb{E} \left[ \left| \widehat{\Phi}(t) - \widehat{\Phi}(0) - \int_0^t \Pi' (\widehat{\Phi}(s) - \nu) ds \right|^2 \right] \\ &\leq \frac{2}{t} \mathbb{E} \left[ |\widehat{\Phi}(t) - \widehat{\Phi}(0)|^2 \right] + \frac{2}{t} \mathbb{E} \left[ \left| \int_0^t \Pi' (\widehat{\Phi}(s) - \nu) ds \right|^2 \right], \\ &\leq \frac{2}{t} + \frac{2}{t} E \int_0^t \int_0^t \text{tr} \{ \Pi' \Pi (\widehat{\Phi}(r) - \nu) (\widehat{\Phi}(s) - \nu)' \} dr ds. \end{aligned} \quad (A.3)$$

Consider the symmetric matrix

$$G(r, s) = (g_{ij}(r, s)) = \mathbb{E}[(\widehat{\Phi}(r) - \nu)(\widehat{\Phi}(s) - \nu)'].$$

One can observe that

$$\begin{aligned}
g_{ij}(r, s) &= \mathbb{E}[(\widehat{\Phi}(r) - \nu)_i(\widehat{\Phi}'(s) - \nu')_j], \\
&= \mathbb{E}[(\mathbb{E}[\mathbf{1}_{\{\alpha(r)=i\}}|\mathcal{F}_r^X] - \nu_i)(\mathbb{E}[\mathbf{1}_{\{\alpha(s)=j\}}|\mathcal{F}_s^X] - \nu_j)], \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\alpha(r)=i\}}|\mathcal{F}_r^X]\mathbb{E}[\mathbf{1}_{\{\alpha(s)=j\}}|\mathcal{F}_s^X]] \\
&\quad - \nu_j\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\alpha(r)=i\}}|\mathcal{F}_r^X]] \\
&\quad - \nu_i\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\alpha(s)=j\}}|\mathcal{F}_s^X]] + \nu_i\nu_j, \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\alpha(r)=i\}}|\mathcal{F}_r^X]\mathbb{E}[\mathbf{1}_{\{\alpha(s)=j\}}|\mathcal{F}_s^X]] - \nu_j\mathbb{P}(\alpha(r) = i) \\
&\quad - \nu_i\mathbb{P}(\alpha(s) = j) + \nu_i\nu_j.
\end{aligned} \tag{A.4}$$

Note also by the Fubini Theorem that

$$\begin{aligned}
&\frac{1}{t} \int_0^t \int_0^t g_{ij}(r, s) dr ds \\
&= \frac{1}{t} \left( \int_0^t \int_r^t g_{ij}(r, s) dr ds + \int_0^t \int_0^r g_{ij}(r, s) dr ds \right), \\
&= \frac{1}{t} \int_0^t \left( \int_r^t g_{ij}(r, s) ds \right) dr + \frac{1}{t} \int_0^t \left( \int_s^t g_{ij}(r, s) dr \right) ds, \\
&= g_1(t) + g_2(t) = 2g_1(t).
\end{aligned}$$

We have the decomposition

$$g_1(t) = h_1(t) + \ell_1(t)$$

where

$$\begin{aligned}
h_1(t) &= \frac{1}{t} \int_0^t \left( \int_r^t h(r, s) ds \right) dr, \\
\ell_1(t) &= \frac{1}{t} \int_0^t \left( \int_r^t \nu_i(\nu_j - \mathbb{P}(\alpha(s) = j)) ds \right) dr,
\end{aligned}$$

with

$$h(r, s) = \mathbb{P}(\alpha(r) = i)\mathbb{P}(\alpha(s) = j|\alpha(r) = i) - \nu_j\mathbb{P}(\alpha(r) = i).$$

Before proceeding further, let us first note the following mixing properties regarding the Markov chain  $\alpha(t)$ . For all  $t \geq 0$  and  $s \leq t$ , denote

$$\begin{aligned}
p(t) &= (\mathbb{P}(\alpha(t) = 1), \dots, \mathbb{P}(\alpha(t) = m))' \in \mathbb{R}^m, \\
P(t, s) &= ((\mathbb{P}(\alpha(t) = j|\alpha(s) = i), i, j \in \mathcal{M}) \in \mathbb{R}^{m \times m},
\end{aligned}$$

which are the probability vector and transition matrix of the Markov chain  $\alpha(t)$ , respectively. Since  $\alpha(t)$  is irreducible, it is ergodic. Consequently, as  $t$  goes to infinity, for the solution of the system

$$\begin{cases} \frac{dp(t)}{dt} = \Pi' p(t) \\ p(0) = p_0 \end{cases} \tag{A.5}$$

satisfying

$$p_{0,i} \geq 0 \quad \text{and} \quad \sum_{i=1}^m p_{0,i} = 1,$$

one can find two positive constants  $\kappa$  and  $K$  such that  $p(t) \rightarrow \nu$  and

$$|p(t) - \nu| \leq K \exp(-\kappa t) \tag{A.6}$$

By virtue of (A.6), it is easily seen that

$$\begin{aligned}
|\ell_1(t)| &= \left| \frac{\nu_i}{t} \int_0^t \left( \int_r^t (\nu_j - \mathbb{P}(\alpha(s) = j)) ds \right) dr \right|, \\
&\leq \frac{\nu_i}{t} \int_0^t \left( \int_r^t |\nu_j - \mathbb{P}(\alpha(s) = j)| ds \right) dr, \\
&\leq \frac{\nu_i K}{t} \int_0^t \left( \int_r^t \exp(-\kappa s) ds \right) dr, \\
&\leq \frac{\nu_i K}{\kappa t} \int_0^t \exp(-\kappa r) dr, \\
&\leq \frac{\nu_i K}{\kappa^2 t}.
\end{aligned} \tag{A.7}$$

Consequently,  $\ell_1(t)$  goes to zero as  $t$  tends to infinity. Next, we shall show that  $h_1(t)$  is bounded. As before, the solution of the system

$$\begin{cases} \frac{\partial P(t, s)}{\partial t} = \Pi' P(t, s) \\ P(s, s) = I_m \end{cases} \tag{A.8}$$

with  $s \leq t$ , also satisfies for two positive constants  $\lambda$  and  $K$ ,  $P(t, s) \rightarrow \mathbf{1}\nu'$  and

$$|P(t, s) - \mathbf{1}\nu'| \leq K \exp(-\lambda(t - s)). \tag{A.9}$$

It follows from (A.9) that

$$\begin{aligned}
|h_1(t)| &= \left| \frac{1}{t} \int_0^t \mathbb{P}(\alpha(r) = i) \left( \int_r^t (\mathbb{P}(\alpha(s) = j|\alpha(r) = i) - \nu_j) ds \right) dr \right| \\
&\leq \frac{1}{t} \int_0^t \left( \int_r^t |\mathbb{P}(\alpha(s) = j|\alpha(r) = i) - \nu_j| ds \right) dr \\
&\leq \frac{K}{t} \int_0^t \left( \int_r^t \exp(-\lambda(s - r)) ds \right) dr \\
&\leq \frac{K}{\lambda t} \int_0^t dr \leq \frac{K}{\lambda}.
\end{aligned}$$

Therefore,  $h_1(t)$  as well as  $g_1(t)$  are bounded sequences which ensures that for some positive constant  $K$  independent of  $t$

$$\left| \mathbb{E} \left[ \frac{1}{t} \int_0^t \int_0^t \text{tr} \{ \Pi' \Pi (\widehat{\Phi}(r) - \nu)(\widehat{\Phi}'(s) - \nu') \} dr ds \right] \right| \leq K. \tag{A.10}$$

Finally, (A.2) together with (A.3) and (A.10) imply (4.6) which completes the proof of Proposition 4.3.  $\square$

## APPENDIX B

We shall now focus on the proof of Proposition 4.5. First of all, we know that

$$\sup_{t \geq 0} |\Pi' \widehat{\Phi}(t)| \leq 1 \quad \text{a.s.}$$

In addition, we also have  $|\widehat{\Phi}(t)| \leq 1$  a.s. Consequently, it follows from (A.1) that

$$|N(t)| \leq |\widehat{\Phi}(t) - \widehat{\Phi}(0)| + \left| \int_0^t \Pi' \widehat{\Phi}(s) ds \right| \leq 1 + t \quad \text{a.s.} \tag{B.1}$$

For each  $i \in \mathcal{M}$ , denote

$$N_i(t) = \int_0^t \sum_{j=1}^n [D(\widehat{\Phi}(s)C(X(s)))']_{ij} dV_j(s)$$

where  $[D(\widehat{\Phi}(s)C(X(s)))']_{ij}$  is the  $ij$ th entry of the matrix  $D(\widehat{\Phi}(s)C(X(s)))'$  and  $V_j(s)$  stands the  $j$ th component of  $V(s)$ . It follows from the well-known Doob's martingale inequality given for example in [18, Theorem 1.7.4, p. 44] that for each  $i \in \mathcal{M}$  and each positive integer  $n$ ,

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq n} \left| \int_0^t N_i(t) - |(D(\widehat{\Phi}(s)C(X(s)))')_{i,\cdot}|^2 ds \right| \geq \log n\right) \\ \leq \frac{1}{n^2}, \end{aligned} \tag{B.2}$$

where  $(D(\widehat{\Phi}(s)C(X(s)))')_{i,\cdot}$  denotes the row vector in the  $i$ th row of the matrix  $D(\widehat{\Phi}(s)C(X(s)))'$ . Hence, we deduce from the Borel-Cantelli Lemma that for almost all  $\omega \in \Omega$ , there is a  $K_1 = K_1(\omega) > 1$  such that for all  $n \geq K_1$  and  $t \leq n$

$$\begin{aligned} \int_0^t |(D(\widehat{\Phi}(s)C(X(s)))')_{i,\cdot}|^2 ds \leq \log n + N_i(t) \text{ a.s.} \\ \leq \log n + 1 + t \text{ a.s.} \end{aligned} \tag{B.3}$$

The last line above follows from (B.1). Dividing both sides of (B.3) by  $t$ , we obtain that for  $n \geq K_2$ ,  $n - 1 \leq t \leq n$ , so

$$\begin{aligned} \frac{1}{t} \int_0^t |(D(\widehat{\Phi}(s)C(X(s)))')_{i,\cdot}|^2 ds \\ \leq \frac{1}{n-1} (\log n + 1 + t) \text{ a.s.} \\ \leq \frac{1}{n-1} (\log n + 1 + n) \text{ a.s.} \\ \leq K_3 \text{ a.s.} \end{aligned} \tag{B.4}$$

and the bound  $K_3$  is independent of  $t$ . Consequently, for some positive constant  $K$  independent of  $t$ , the quadratic variation of the martingale is bounded by  $Kt$  almost surely. That is, (B.4) implies that  $\langle N, N \rangle_t \leq Kt$  a.s. Finally, we deduce from the strong law of large numbers for local martingales [15] that

$$\lim_{t \rightarrow \infty} \frac{1}{t} N(t) = 0 \text{ a.s.}$$

which concludes the proof of Proposition 4.5.  $\square$

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