*Studia Scientiarum Mathematicarum Hungarica*
DOI: 10.1556/SScMath.2007.1047

# A MOMENT APPROACH FOR THE ALMOST SURE CENTRAL LIMIT THEOREM FOR MARTINGALES

B. BERCU and J. C. FORT

Laboratoire de Statistique et Probabilités, UMR C5583, Université Paul Sabatier,
118 Route de Narbonne, 31062 Toulouse Cedex, France
e-mail: bercu@math.ups-tlse.fr, fort@math.ups-tlse.fr

*Communicated by E. Csáki*

(Received January 4, 2006; accepted April 18, 2006)

**Abstract**

We prove the almost sure central limit theorem for martingales via an original approach which uses the Carleman moment theorem together with the convergence of moments of martingales. Several statistical applications to autoregressive and branching processes are also provided.

## 1. Introduction.

Let $(X_n)$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_n] = 0$, $\mathbb{E}[X_n^2] = \sigma^2$. The almost sure central limit theorem (ASCLT) associated with $(X_n)$ states that the empirical measure

$$G_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{S_k/\sqrt{k}} \quad \text{with} \quad S_n = \sum_{k=1}^{n} X_k$$

converges a.s. to the standard $\mathcal{N}(0, \sigma^2)$ distribution. It was simultaneously established by Brosambler [4] and Schatte [19], [20] and in the present form by Lacey and Phillip [13]. The most achieved result on ASCLT for independent random variables was obtained by Berkes and Csáki in [1]. While a wide literature concerning the ASCLT for independent random variables is

2000 *Mathematics Subject Classification.* 60G42, 60F05.
*Key words and phrases.* Martingale transforms, moments.

Author: please give full name.

now available, very few references may be found on the ASCLT for martingales apart from the important contribution of Chaabane *et al.* [5], [6], [7] and Lifshits [16], [17]. Let $(\varepsilon_n)$ be a martingale difference sequence adapted to an appropriate filtration $\mathbb{F} = (\mathcal{F}_n)$ with $\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] = \sigma^2$ a.s. and denote by $(\phi_n)$ a sequence of random variables adapted to $\mathbb{F}$. We shall investigate the ASCLT for the real martingale transform $(M_n)$ given by

$$M_n = \sum_{k=1}^{n} \phi_{k-1}\varepsilon_k.$$

The explosion coefficient associated with $(\phi_n)$

$$f_n = \frac{\phi_n^2}{s_n} \qquad \text{where} \qquad s_n = \sum_{k=0}^{n} \phi_k^2$$

will play a crucial role in all the sequel. Hereafter, we assume that $(s_n)$ increases a.s. to infinity. One of the most accurate ASCLT for martingales, due to Chaabane [5], is as follows.

THEOREM 1. *Let $\Delta M_n = M_n - M_{n-1}$ and denote by $(V_n)$ a positive predictable sequence such that*

$$\lim_{n\to\infty} V_n^{-2} s_{n-1} = 1 \qquad a.s. \tag{1.1}$$

$$\text{For all } \varepsilon > 0 \qquad \sum_{n=1}^{\infty} V_n^{-2}\mathbb{E}\big[\Delta M_n^2 \mathbb{I}_{(|\Delta M_n| > \varepsilon V_n)} \mid \mathcal{F}_{n-1}\big] < \infty \qquad a.s. \tag{1.2}$$

$$\text{For some } a > 0 \quad \sum_{n=1}^{\infty} V_n^{-2a}\mathbb{E}\big[|\Delta M_n|^{2a} \mathbb{I}_{(|\Delta M_n| \leqq V_n)} \mid \mathcal{F}_{n-1}\big] < \infty \quad a.s. \tag{1.3}$$

*Then, $(M_n)$ satisfies the ASCLT*

$$\frac{1}{\log V_n^2} \sum_{k=1}^{n} \left( \frac{V_{k+1}^2 - V_k^2}{V_{k+1}^2} \right) \delta_{M_k/V_k} \Longrightarrow G \quad a.s.$$

*where $G$ stands for the standard $\mathcal{N}(0, \sigma^2)$ distribution.*

One can easily check that, under the assumptions of Theorem 1, $V_{n+1}^2$ is a.s. equivalent to $V_n^2$ so that the explosion coefficient $f_n$ tends to zero a.s. In addition, the simple choice $V_n^2 = s_{n-1}$ leads to the weak almost sure convergence of the empirical measure

$$G_n = \frac{1}{\log s_n} \sum_{k=1}^{n} f_k \delta_{M_k/\sqrt{s_{k-1}}}$$

to $G$. In other words, for any bounded continuous real function $h$

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k h\left(\frac{M_k}{\sqrt{s_{k-1}}}\right) = \int_{\mathbb{R}} h(x), dG(x) \quad \text{a.s.} \tag{1.4}$$

In all what follows, we shall say that $(M_n)$ satisfies a polynomial almost sure central limit theorem (PASCLT) if convergence (1.4) holds for any polynomial function $h$ over $\mathbb{R}$. One can observe that a PASCLT implies a standard ASCLT, whenever the limiting distribution is caracterized by its moments. As a matter of fact, the boundeness of the moments ensures the tightness of $G_n$. Hence, all the moments converge and the limiting value is unique.

One might wonder if the theoretical study of ASCLT for martingales is completely acheived by Theorem 1. To be more precise, is it possible to characterize the largest class of real martingale transforms satisfying the ASCLT ? As noticed by Lifshits [16], the assumptions of Theorem 1 are too restrictive. For example, (1.2) is not satisfied for martingales with rare jumps of magnitude greater than $V_n$ as (1.2) immediately implies that, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{1}_{(|\Delta M_n| > \varepsilon V_n)} < \infty \qquad \text{a.s.}$$

Moreover, one can realize that (1.3) does not hold for martingales with explosion coefficient $f_n$ decreasing slowly to zero. More precisely, assume that $(\varepsilon_n)$ is a martingale difference sequence such that $\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] = 1$ a.s. For example, we can set $\xi_n = (\varepsilon_1, \ldots, \varepsilon_n)$ and choose

$$\varepsilon_{n+1} = A_{n+1} \mathbb{1}_{\|\xi_n\| \leqq c} + B_{n+1} \mathbb{1}_{\|\xi_n\| > c}$$

where $(A_n)$, $(B_n)$ are two sequences of centered, independent random variables with variance 1, having moments of all orders, and $c$ is a positive constant. Let $(\phi_n)$ be positive deterministic such that $\phi_0 = 1$ and for all $n \geqq 1$

$$\phi_n^2 = \frac{1}{\log(e+n)} \prod_{k=1}^{n} \frac{\log(e+k)}{\log(e+k) - 1}. \tag{1.5}$$

Then, $(s_n)$ increases to infinity, $f_n$ tends to zero almost surely as

$$s_n = \prod_{k=1}^{n} \frac{\log(e+k)}{\log(e+k)-1} \quad \text{and} \quad f_n = \frac{1}{\log(e+n)}.$$

However, (1.3) always fails as it reduces to

$$\sum_{n=0}^{\infty} f_n^a = \infty \quad \text{a.s.}$$

Nevertheless, we will show in the sequel that $(M_n)$ satisfies an ASCLT.

The paper is organized as follows. In section 2, we establish a new ASCLT based on the Carleman moment Theorem together with the convergence of moments of martingales. Section 3 is devoted to similar results when the explosion coefficient $f_n$ converges a.s. to a positive random variable. Statistical applications to autoregressive and branching processes are developed in section 4, while all technical proofs are postponed in the Appendices.

## 2. On Carleman approach

The classical moment problem concerns the question whether or not a given sequence of moments $(m_n)$ uniquely determines the associated probability distribution. One can find many probability distributions which are not uniquely determined by their moments, for example the log-normal distribution. However, the celebrated Carleman Theorem (see *e.g.* [9] page 228) gives a positive answer to that question under a suitable condition on the moments $(m_n)$.

Theorem 2. *A probability distribution is uniquely determined by its moments $(m_n)$ if*

$$\sum_{n=1}^{\infty} m_{2n}^{-1/2n} = \infty. \tag{2.1}$$

We will make use of this result in the martingale framework via the following theorem where the first convergence (2.3) was recently proven in [2].

Theorem 3. *Assume that $(\varepsilon_n)$ is a martingale difference sequence such that $\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] = \sigma^2$ a.s. and satisfying, for some integer $p \geqq 1$ and some real $a > 2p$, the moment condition*

$$\sup_{n \geqq 0} \mathbb{E}\big[|\varepsilon_{n+1}|^a \mid \mathcal{F}_n\big] < \infty \quad a.s. \tag{2.2}$$

*In addition, assume that the explosion coefficient* $f_n$ *tends to zero a.s. Then*

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k \left( \frac{M_k}{\sqrt{s_{k-1}}} \right)^{2p} = \frac{\sigma^{2p}(2p)!}{2^p p!} \quad a.s. \tag{2.3}$$

*In addition, we also have*

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k \left( \frac{M_k}{\sqrt{s_{k-1}}} \right)^{2p-1} = 0 \quad a.s. \tag{2.4}$$

The proof of this theorem is postponed to Appendix A.

One can observe that the Gaussian limit distribution clearly satisfies Carleman's moment condition (2.1). Combining the last two theorems, we deduce the following ASCLT for martingales.

COROLLARY 4. *Assume that* $(\varepsilon_n)$ *is a martingale difference sequence such that* $\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] = \sigma^2$ *a.s. and satisfying, for all integer* $p \geqq 1$,

$$\sup_{n\geqq 0} \mathbb{E}\big[|\varepsilon_{n+1}|^p \mid \mathcal{F}_n\big] < \infty \quad a.s. \tag{2.5}$$

*In addition, assume that the explosion coefficient* $f_n$ *tends to zero a.s. Then, the martingale transform* $(M_n)$ *satisfies the ASCLT.*

Consider once again the enlightening example of the introduction. The moment condition (2.5) clearly holds for all integer $p \geqq 1$ since $(A_n)$ and $(B_n)$ have moments of all orders. Moreover, $f_n$ decreases to zero a.s. with a logarithmic rate of convergence. Consequently, $(M_n)$ satisfies the ASCLT given by (1.4).

## 3. Extension to explosive martingales

One might wonder whether or not an ASCLT holds when $f_n$ converges a.s. to a positive random variable $f$. Our goal is now to show that this is the case. First of all, we need an asymptotic result for the moments similar to that of Theorem 3. For any integer $p \geqq 1$, set

$$\sigma_n(p) = \mathbb{E}[\varepsilon_{n+1}^p \mid \mathcal{F}_n].$$

THEOREM 5. *Assume that* $(\varepsilon_n)$ *is a martingale difference sequence such that* $\mathbb{E}\big[\varepsilon_{n+1}^2 \mid \mathcal{F}_n\big] = \sigma^2$ *a.s. and satisfying, for some integer* $p \geqq 1$, *the moment condition* (2.2). *In addition, suppose that for any* $2 \leqq q \leqq 2p$,

$$\lim_{n\to\infty} \sigma_n(q) = \sigma(q) \quad a.s. \tag{3.1}$$

*where $\sigma(q) = 0$ if $q$ is odd. Moreover, assume that the explosion coefficient $f_n$ converges a.s. to a random variable $f$ with $0 < f < 1$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{M_k}{\sqrt{s_{k-1}}} \right)^{2p} = l(p, f) \quad a.s. \tag{3.2}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{M_k}{\sqrt{s_{k-1}}} \right)^{2p-1} = 0 \quad a.s. \tag{3.3}$$

*where $l(0, f) = 1$ and, for $p \geqq 1$, $l(p, f)$ satisfies the recurrence equation*

$$l(p, f) = \frac{1}{1 - (1 - f)^p} \sum_{k=1}^{p} C_{2p}^{2k} f^k (1 - f)^{p-k} \sigma(2k) l(p - k, f). \tag{3.4}$$

The proof of Theorem 5 is postponed to Appendix B.

We now propose a non Gaussian ASCLT for explosive martingales.

THEOREM 6. *Assume that $(\varepsilon_n)$ is a martingale difference sequence such that $\mathbb{E}\big[\varepsilon_{n+1}^2 \mid \mathcal{F}_n\big] = \sigma^2$ a.s. and satisfying, for all integer $p \geqq 1$, the moment conditions (2.5) and (3.1). In addition, assume that the explosion coefficient $f_n$ converges a.s. towards a random variable $f$ with $0 < f < 1$, and that the sequence $\big(l(p, f)\big)$ satisfies Carleman's moment condition. Then, there exists a unique probability distribution $\mathcal{H}_f$ such that*

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{M_k/\sqrt{s_{k-1}}} \Longrightarrow \mathcal{H}_f \quad a.s. \tag{3.5}$$

*Moreover, if the limiting moments sequence $\big(\sigma(p)\big)$ defines a probability distribution with Laplace transform $L_\sigma$ finite on a neighborhood of the origin, then the Laplace transform $L_{\mathcal{H}_f}$ of $\mathcal{H}_f$ exists a.s. on a neighborhood of the origin and is given by*

$$L_{\mathcal{H}_f}(t) = \prod_{k=0}^{\infty} L_\sigma\big( f^{1/2} (1 - f)^{k/2} t \big) \quad a.s. \tag{3.6}$$

REMARK 7. On the one hand, an easy sufficient condition which ensures that the sequence $\big(l(p, f)\big)$ satisfies Carleman's moment condition is that there exists some constant $C > 0$ such that

$$\sigma(2p) = O(C^p p^{2p}).$$

On the other hand, one can see that (3.5) holds for any polynomial function. In addition, if all the moments $\sigma(2p)$ coincide with those of an $\mathcal{N}(0,\sigma^2)$ random variable, then $\mathcal{H}_f$ is simply the $\mathcal{N}(0,\sigma^2)$ distribution.

Finally, set $r=(1-f)^{-1/2}$ and assume that $r$ is an integer. From equation (3.6), it follows that $\mathcal{H}_f$ has the same distibution as

$$\left(1-\frac{1}{r^2}\right)^{1/2}\sum_{k=0}^{\infty}\frac{\xi_k}{r^k}$$

where the $\xi_k$ are independent random variables with moments $\sigma(p)$. Let $(B_n)$ be a sequence of independent random variables uniformly distributed over the set $\{0,1,\dots,r-1\}$. If we choose $\xi_k=2B_k-(r-1)$, then $\mathcal{H}_f$ coincides with the uniform distribution on the interval $\big[-(r^2-1)^{1/2},(r^2-1)^{1/2}\big]$ (see e.g. [8] page 44). As a matter of fact, $\mathcal{H}_f$ has the same distribution as

$$\left(1-\frac{1}{r^2}\right)^{1/2}\sum_{k=0}^{\infty}\frac{\xi_k}{r^k}=\left(1-\frac{1}{r^2}\right)^{1/2}\left(2\sum_{k=0}^{\infty}\frac{B_k}{r^k}-r\right).$$

Proof of Theorem 6. We obtain convergence (3.5) proceeding exactly as in the proof of Corollary 4. Hence, it only remains to prove relation (3.6). We introduce the following Laplace transforms or moment generating functions as extended real numbers

$$L_{\mathcal{H}_f}(t)=\sum_{p=0}^{\infty}\frac{l(p,f)}{(2p)!}t^{2p}\quad\text{and}\quad L_\sigma(t)=\sum_{p=0}^{\infty}\frac{\sigma(2p)}{(2p)!}t^{2p}.$$

One can observe that if $L_\sigma$ is finite on a neighborhood of the origin, then $\sigma(2p)=O(C^pp^{2p})$ for some constant $C>0$. Then, we easily deduce from equation (3.4) that $l(p,f)=O(D^pp^{2p})$ for some other constant $D>0$ which yields the existence of $L_{\mathcal{H}_f}$ on a neighborhood of zero. Using again formula (3.4), we obtain that

$$\begin{aligned}L_{\mathcal{H}_f}(t)&=\sum_{p=0}^{\infty}\frac{t^{2p}}{(2p)!}\sum_{k=0}^{p}C_{2p}^{2k}f^k(1-f)^{p-k}\sigma(2k)l(p-k,f),\\&=\sum_{k=0}^{\infty}\frac{\sigma(2k)}{(2k)!}f^kt^{2k}\sum_{p=k}^{\infty}\frac{l(p-k,f)}{\big(2(p-k)\big)!}(1-f)^{p-k}t^{2(p-k)},\end{aligned}$$

$$= L_\sigma(f^{1/2}t) \sum_{p=0}^{\infty} \frac{l(p,f)}{(2p)!}\big((1-f)^{1/2}t\big)^p,$$

$$= L_\sigma(f^{1/2}t) L_{\mathcal{H}_f}\big((1-f)^{1/2}t\big),$$

which immediately leads to (3.6), completing the proof of Theorem 6. □

## 4. Applications

### 4.1. Linear regression

Consider the stochastic linear regression model given by

$$X_n = \theta\phi_{n-1} + \varepsilon_n \tag{4.1}$$

where $X_n$ and $\phi_n$ are the observation and the regression variable, respectively. We assume that the noise $(\varepsilon_n)$ is a martingale difference sequence such that $\mathbb{E}\big[\varepsilon_{n+1}^2 \mid \mathcal{F}_n\big] = \sigma^2$ a.s. In order to estimate the unknown real parameter $\theta$, we shall make use of the least squares estimator $\widehat{\theta}_n$. By definition, $\widehat{\theta}_n$ minimizes the mean square error

$$\sum_{k=1}^{n} (X_k - \theta\phi_{k-1})^2.$$

Setting $s_n = \displaystyle\sum_{k=0}^{n} \phi_k^2$, a straightforward computation yields

$$\widehat{\theta}_n = \frac{1}{s_{n-1}} \sum_{k=1}^{n} \phi_{k-1} X_k.$$

Then, it follows from (4.1) that

$$\widehat{\theta}_n - \theta = \frac{1}{s_{n-1}} \sum_{k=1}^{n} (\phi_{k-1}X_k - \phi_{k-1}^2\theta) = \frac{M_n}{s_{n-1}}$$

where

$$M_n = \sum_{k=1}^{n} \phi_{k-1}\varepsilon_k.$$

A direct application of Corollary 4 is as follows.

COROLLARY 8. *Assume that* $(\varepsilon_n)$ *is a martingale difference sequence satisfying, for all integer* $p \geqq 1$, *the moment condition* (2.5). *In addition, suppose that* $s_n$ *increases a.s. to infinity and that* $f_n$ *converge a.s. towards zero. Then,* $(\widehat{\theta}_n)$ *satisfies the PASCLT*

$$\frac{1}{\log s_n} \sum_{k=1}^{n} f_k \, \delta_{\sqrt{s_k}(\widehat{\theta}_k - \theta)} \Longrightarrow \mathcal{N}(0, \sigma^2) \quad a.s. \tag{4.2}$$

*More particularly, assume that for some positive constant* $\tau$

$$\lim_{n \to \infty} \frac{s_n}{n} = \tau \quad a.s.$$

*Then, we have the PASCLT*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \, \delta_{\sqrt{k}(\widehat{\theta}_k - \theta)} \Longrightarrow \mathcal{N}\left(0, \frac{\sigma^2}{\tau}\right) \quad a.s. \tag{4.3}$$

REMARK 9. We immediately infer from (4.3) that for all integer $p \geqq 1$

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{p-1} \big(\widehat{\theta}_k - \theta\big)^{2p} = \frac{\sigma^{2p}(2p)!}{\tau^p 2^p p!} \quad \text{a.s.}$$

while

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} k^{p-3/2} \big(\widehat{\theta}_k - \theta\big)^{2p-1} = 0 \quad \text{a.s.}$$

The simple choice $\phi_n = X_n$ in (4.1) leads to the linear autoregressive model

$$X_n = \theta X_{n-1} + \varepsilon_n. \tag{4.4}$$

In the stable case $|\theta| < 1$, it is well-known that $f_n$ tends a.s. to zero and $s_n/n$ converges a.s. to $\sigma^2/(1-\theta^2)$ (see e.g. [8], [14], [21]). Hence, it follows from (4.3) that $(\widehat{\theta}_n)$ satisfies the PASCLT

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \, \delta_{\sqrt{k}(\widehat{\theta}_k - \theta)} \Longrightarrow \mathcal{N}(0, 1-\theta^2) \quad \text{a.s.}$$

In the unstable case $|\theta| = 1$, once again $f_n \to 0$ but $s_n/n^2$ diverges. However, by formula (3.5) of Wei [21], $\log s_n$ is a.s. equivalent to $2\log n$. Consequently, only (4.2) holds replacing $\log s_n$ by $2\log n$. Similarly to Corollary 8, a direct application of Theorem 6 for explosive martingales is as follows.

COROLLARY 10. *Assume that $(\varepsilon_n)$ is a martingale difference sequence satisfying, for all integer $p \geqq 1$, the moment condition (2.5) and (3.1). In addition, assume that the explosion coefficient $f_n$ converges a.s. towards a random variable $f$ with $0 < f < 1$, and that the sequence $\big(l(p,f)\big)$ satisfies Carleman's moment condition. Then, $(\widehat{\theta}_n)$ satisfies the PASCLT*

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{\sqrt{s_{k-1}}(\widehat{\theta}_k-\theta)} \Longrightarrow \mathcal{H}_f \quad a.s. \tag{4.5}$$

*In addition, assume that for some positive random variable $\tau$*

$$\lim_{n\to\infty} (1-f)^n s_n = \tau \quad a.s.$$

*Then, there exists a unique probability distribution $\mathcal{H}_f(\tau)$ such that*

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{(\widehat{\theta}_k-\theta)/(1-f)^{k/2}} \Longrightarrow \mathcal{H}_f(\tau) \quad a.s. \tag{4.6}$$

*Lastly, if all the moments $\sigma(2p)$ coincide with those of an $\mathcal{N}(0,\sigma^2)$ random variable, then we have the PASCLT*

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{(\widehat{\theta}_k-\theta)/(1-f)^{k/2}} \Longrightarrow \mathcal{N}\left(0, \frac{\sigma^2}{\tau(1-f)}\right) \quad a.s.$$

REMARK 11. As (4.6) holds for any polynomial function, We find that for all integer $p \geqq 1$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\big(\widehat{\theta}_k-\theta\big)^{2p}}{(1-f)^{kp}} = \frac{l(p,f)}{\tau^p(1-f)^p} \quad \text{a.s.}$$

while

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\big(\widehat{\theta}_k-\theta\big)^{2p-1}}{(1-f)^{k(p-1/2)}} = 0 \quad \text{a.s.}$$

Consider once again the linear autoregressive model given by (4.4). In the explosive case $|\theta| > 1$, $\theta^{-n}X_n$ converges a.s. and in mean square to the random variable

$$Y = X_0 + \sum_{k=1}^{\infty} \theta^{-k}\varepsilon_k.$$

In addition, it is shown in [15] that $Y \neq 0$ a.s. Hence, it follows from Toeplitz's lemma that $f_n \to (\theta^2 - 1)/\theta^2$ a.s. and $s_n/\theta^{2n}$ converges a.s. to $\theta^2 Y^2/(\theta^2 - 1)$ (see e.g. [8], [14]). Consequently, we deduce from (4.6) that

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{|\theta|^k(\widehat{\theta}_k - \theta)} \Longrightarrow \mathcal{H}_f(\tau) \quad \text{a.s.}$$

with $\tau = \theta^2 Y^2/(\theta^2 - 1)$. More particularly, if all the moments $\sigma(2p)$ coincide with those of an $\mathcal{N}(0, \sigma^2)$ random variable, we have the PASCLT

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{|\theta|^k(\widehat{\theta}_k - \theta)} \Longrightarrow \mathcal{N}\left(0, \frac{\sigma^2(\theta^2 - 1)}{Y^2}\right) \quad \text{a.s.}$$

### 4.2. Branching processes

Consider the Galton–Watson process given by

$$X_n = \sum_{k=1}^{X_{n-1}} Y_{n,k} \tag{4.7}$$

with $X_0 = 1$. The random variable $X_n$ denotes the size of the $n$-th generation while $Y_{n,k}$ is the number of offsprings of the $k$-th individual in the $(n-1)$-th generation. We assume that $(Y_{n,k})$ is a sequence of independent and identically distributed random variables such that $Y_{n,k} \geqq 1$. The distribution of $(Y_{n,k})$, with finite mean $m$ and positive variance $\sigma^2$, is commonly called the offspring distribution. We also suppose that $(Y_{n,k})$ has finite moments of any order. Relation (4.7) can be rewritten as

$$X_n = mX_{n-1} + \xi_n \tag{4.8}$$

where $\xi_n = X_n - \mathbb{E}[X_n \mid \mathcal{F}_{n-1}]$. If

$$\varepsilon_n = \frac{\xi_n}{\sqrt{X_{n-1}}},$$

we clearly have $\mathbb{E}[\varepsilon_{n+1} \mid \mathcal{F}_n] = 0$ and $\mathbb{E}\big[\varepsilon_{n+1}^2 \mid \mathcal{F}_n\big] = \sigma^2$ a.s. The conditional least square estimator of $m$ is given by

$$\widehat{m}_n = \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_{k-1}}.$$

Consequently, we obtain from (4.8) that

$$\widehat{m}_n - m = \frac{M_n}{s_{n-1}} \quad \text{where} \quad M_n = \sum_{k=1}^n \phi_{k-1}\varepsilon_k$$

and $\phi_n = \sqrt{X_n}$. In the supercritical case $m > 1$, it is well-known that $m^{-n}X_n$ converges a.s. and in mean square to the nonzero random variable

$$L = X_0 + \sum_{k=1}^{\infty} m^{-k}\xi_k.$$

Thus, we deduce from Toeplitz's lemma that $f_n \to (m-1)/m$ a.s. and $s_n/m^n$ converges a.s. to $mL/(m-1)$ (see e.g. [10]). Our purpose is now to propose a second application of Theorem 6 to $(\widehat{m}_n)$. Since $(Y_{n,k})$ has finite moments of any order, the same remains true for the sequence $(\varepsilon_n)$. Hence, in order to make use of Theorem 6, it is enough to verify the convergence of the conditional moments associated with $(\varepsilon_n)$. Moreover, it follows from (4.7) that

$$\varepsilon_n = \frac{1}{\sqrt{X_{n-1}}} \sum_{k=1}^{X_{n-1}} (Y_{n,k} - m).$$

Consequently, applying the standard central limit theorem, the distribution of $\varepsilon_{n+1}$ conditionally to $\mathcal{F}_n$ converges to the Gaussian distribution $\mathcal{N}(0, \sigma^2)$. For any $p \geqq 1$, as the moments of order $2p$ of $\varepsilon_{n+1}$ conditionally to $\mathcal{F}_n$ are bounded, a classsical argument of uniform integrability (see e.g. [3], Theorem 25.12) leads to the convergence of these moments to those of the $\mathcal{N}(0, \sigma^2)$ distribution. Therefore, a straightforward application of Theorem 6, similar to Corollary 10, is as follows.

COROLLARY 12. *In the supercritical case $m > 1$, $(\widehat{m}_n)$ satisfies the PASCLT*

$$\frac{1}{n}\sum_{k=1}^n \delta_{\sqrt{s_{k-1}}(\widehat{m}_k - m)} \Longrightarrow \mathcal{N}(0, \sigma^2) \quad a.s. \tag{4.9}$$

*Moreover, we also have*

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{m^{k/2}(\widehat{m}_k-m)} \Longrightarrow \mathcal{N}\left(0, \frac{(m-1)\sigma^2}{L}\right) \quad a.s. \tag{4.10}$$

REMARK 13. A standard ASCLT can be found in [18]. In addition, we infer from (4.10) that for all integer $p \geqq 1$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} m^{kp}(\widehat{m}_k - m)^{2p} = \frac{(m-1)^p\sigma^{2p}(2p)!}{L^p 2^p p!} \quad \text{a.s.}$$

while

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} m^{k(p-1/2)}(\widehat{m}_k - m)^{2p-1} = 0 \quad \text{a.s.}$$

## Appendix A

Appendix A is devoted to the

PROOF OF THEOREM 3. We shall only prove convergence for odd moments (2.4) as convergence for even moments (2.3) was already established in [2]. First of all, for any $p \geqq 1$, set

$$v_n(p) = \frac{\left(\sqrt{s_n}\right)^{2p-1} - \left(\sqrt{s_{n-1}}\right)^{2p-1}}{\left(\sqrt{s_n}\right)^{2p-1}}.$$

As $M_{n+1} = M_n + \phi_n \varepsilon_{n+1}$, we have for any $p \geqq 1$

$$M_{n+1}^{2p-1} = \sum_{k=0}^{2p-1} C_{2p-1}^k \phi_n^k \varepsilon_{n+1}^k M_n^{2p-1-k}. \tag{A.1}$$

Consequently, putting

$$V_n = \left(\frac{M_n}{\sqrt{s_{n-1}}}\right)^{2p-1}$$

we easily deduce from (A.1) that for any $n \geqq 1$

$$V_{n+1} + \mathcal{A}_n = V_1 + \mathcal{B}_{n+1} + \mathcal{W}_{n+1} \tag{A.2}$$

where

$$\mathcal{A}_n = \sum_{k=1}^{n} v_k(p) V_k, \qquad \mathcal{W}_{n+1} = \sum_{k=1}^{n} s_k^{-p} \sqrt{s_k} \phi_k^{2p-1} \varepsilon_{k+1}^{2p-1},$$

$$\mathcal{B}_{n+1} = \sum_{l=1}^{2p-2} C_{2p-1}^{l} B_{n+1}(l), \qquad B_{n+1}(l) = \sum_{k=1}^{n} \varphi_k(l) \varepsilon_{k+1}^{l},$$

and for any $1 \leqq l \leqq 2(p-1)$, $\varphi_k(l) = s_k^{-p}\sqrt{s_k} M_k^{2p-1-l}\phi_k^l$. Via a standard truncation argument, we may assume without loss of generality that each $\varphi_k(l)$ is a bounded random variable. Hereafter, by use of (A.2), we are in position to prove convergence (2.4) by induction on the power $p \geqq 1$. For $p = 1$, the term $\mathcal{B}_{n+1}$ in the right-hand side of (A.2) vanishes. In addition, it is well-known from [2], [8] or [14] that $M_n^2 = O(s_{n-1} \log s_{n-1})$ a.s. so that $V_{n+1} = o(\log s_n)$ a.s. Moreover, $(\mathcal{W}_n)$ is a locally square integrable martingale with increasing process

$$\langle \mathcal{W} \rangle_{n+1} = \sigma^2 \sum_{k=1}^{n} f_k.$$

By the elementary inequality $x \leqq -\log(1-x)$ for $0 < x < 1$, we have for all $n \geqq 1$, $f_n \leqq -\log(1-f_n)$ so that $f_n \leqq \log s_n - \log s_{n-1}$ which implies that $\langle \mathcal{W} \rangle_{n+1} \leqq \sigma^2 \log s_n$. Hence, we deduce from the standard strong law of large numbers for martingales that $\mathcal{W}_{n+1} = o(\log s_n)$ a.s. Consequently, it immediately follows from (A.2) that $\mathcal{A}_n = o(\log s_n)$ a.s. However, we clearly have $f_n = a_n(1) v_n(1)$ with

$$a_n(1) = \frac{\sqrt{s_n} + \sqrt{s_{n-1}}}{\sqrt{s_n}}. \tag{A.3}$$

As $a_n(1) \to 2$ and $\mathcal{A}_n = o(\log s_n)$ a.s. it is not hard to see that a.s.

$$\sum_{k=1}^{n} f_k V_k = o(\log s_n) + o(T_n) \quad \text{with} \quad T_n = \sum_{k=1}^{n} v_k(1) |V_k|. \tag{A.4}$$

Moreover, via the Cauchy–Schwarz inequality

$$T_n^2 \leqq \sum_{k=1}^{n} f_k \sum_{k=1}^{n} f_k V_k^2$$

because, for all $n \geqq 1$, $a_n(1) \geqq 1$ so $v_n(1) \leqq f_n$. Furthermore, we can deduce from convergence (2.3) with $p = 1$ that

$$\sum_{k=1}^{n} f_k V_k^2 = O(\log s_n) \qquad \text{a.s.}$$

Consequently, $T_n = O(\log s_n)$ a.s. which, by use of (A.4), clearly leads to (2.4) for $p = 1$. Now, let $p \geqq 2$ and assume that convergence (2.4) holds for any power $q$ with $1 \leqq q \leqq p - 1$. We infer from formula (2.5) of [2] or formula (2.30) of [21] that $M_n^{2p} = O(s_{n-1}^p \log s_{n-1})$ a.s. so that $V_{n+1} = o(\log s_n)$ a.s. Next, we may assume without loss of generality that for all $n \geqq 0$, $\sigma_n(2p) \leqq C$ a.s. for some constant $C \geqq 1$. On the one hand, it follows from Chow's lemma (see e.g. Duflo [8] Theorem 1.3.18 p. 22) that

$$\mathcal{W}_{n+1} = o\Big(\sum_{k=1}^{n} f_k^p\Big) + O\Big(\sum_{k=1}^{n} f_k^{p-1/2}\Big) \quad \text{a.s.}$$

Hence, as $f_n \leqq 1$ and $f_n \to 0$ a.s., we find that

$$\mathcal{W}_{n+1} = o(\log s_n) \qquad \text{a.s.} \tag{A.5}$$

On the other hand, we also claim that

$$\mathcal{B}_{n+1} = o(\log s_n) \qquad \text{a.s.} \tag{A.6}$$

In order to prove (A.6), it is only necessary to show that for any integer $1 \leqq l \leqq 2(p-1)$, $B_{n+1}(l) = o(\log s_n)$ a.s. We can split $B_{n+1}(l)$ into two terms, $B_{n+1}(l) = C_{n+1}(l) + D_n(l)$ where

$$C_{n+1}(l) = \sum_{k=1}^{n} \varphi_k(l) e_{k+1}(l) \quad \text{and} \quad D_n(l) = \sum_{k=1}^{n} \varphi_k(l)\sigma_k(l)$$

with, for any $1 \leqq l \leqq 2(p-1)$, $e_{k+1}(l) = \varepsilon_{k+1}^l - \sigma_k(l)$. First, for any $1 \leqq l \leqq p$, the sequence $\big(C_n(l)\big)$ is a locally square integrable martingale satisfying via the strong law of large numbers for martingales $\big|C_{n+1}(l)\big|^2 = O\big(\tau_n(l) \log \tau_n(l)\big)$ a.s. where

$$\tau_n(l) = \sum_{k=1}^{n} \big|\varphi_k(l)\big|^2.$$

Moreover, one can easily deduce from formulas (2.5) and (2.6) of [2] that $\tau_n(l) = O\big((\log s_n)^d\big)$ a.s. with $d = 2(p-1)/p$. Consequently, as $d < 2$, we immediately obtain that for any $1 \leqq l \leqq p$,

$$C_{n+1}(l) = o(\log s_n) \qquad \text{a.s.} \tag{A.7}$$

Next, for $p \geqq 3$ and for any $p+1 \leqq l \leqq 2(p-1)$, we find via Chow's lemma that either $\big(C_n(l)\big)$ converges a.s. or $C_{n+1}(l) = o\big(\nu_n(l)\big)$ a.s. where

$$\nu_n(l) = \sum_{k=1}^{n} \big|\varphi_k(l)\big|^{\delta} \leqq \sum_{k=1}^{n} f_k^p \left(\frac{M_k^2}{s_{k-1}}\right)^{\rho}$$

with $\delta = 2p/l$ and $2\rho = p(\delta-1) - \delta$. Since $p \geqq 3$, we obviously have $1 < \delta < 2$ and $0 < \rho < p$. In addition, it follows from the Hölder inequality that

$$\nu_n(l) \leqq \left(\sum_{k=1}^{n} f_k^p\right)^{1-\rho/p} \left(\sum_{k=1}^{n} f_k^p \left(\frac{M_k^2}{s_{k-1}}\right)^{p}\right)^{\rho/p}.$$

Hence, as $f_n \to 0$ a.s., we infer from (2.3) that $\nu_n(l) = o(\log s_n)$ so that (A.7) holds fo any $1 \leqq l \leqq 2(p-1)$. In order to prove (A.6), as $D_n(1) = 0$, it remains to show that for any $2 \leqq l \leqq 2(p-1)$

$$D_n(l) = o(\log s_n) \qquad \text{a.s.} \tag{A.8}$$

One can easily see from the Hölder inequality that for each $2 \leqq l \leqq 2(p-1)$, $\big|\sigma_n(l)\big| \leqq C$ a.s. Consequently, we find that for any $2 \leqq l \leqq 2(p-1)$

$$\big|D_n(l)\big| \leqq C \sum_{k=1}^{n} f_k^{l/2} \left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2p-1-l} \qquad \text{a.s.} \tag{A.9}$$

We shall study the asymptotic behavior of $D_n(l)$ in the three following cases for proving (A.8).

**Case 1.** Let $l = 2$. It follows from the induction assumption that for any integer $1 \leqq q \leqq p-1$

$$\sum_{k=1}^{n} f_k \left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2q-1} = o(\log s_n) \qquad \text{a.s.} \tag{A.10}$$

By use of (A.10) with $q = p-1$, we obtain that

$$D_n(2) = \sigma^2 \sum_{k=1}^{n} f_k (1-f_k)^{p-3/2} \left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2p-3} = o(\log s_n) \quad \text{a.s.}$$

**Case 2.** Assume that $4 \leqq l \leqq 2(p-1)$ with $l$ even. If $2 \leqq p \leqq 5$, we proceed exactly as in b). Next, if $p \geqq 6$, we have to consider two cases.

a) If $4 \leqq l \leqq p-2$ with $l$ even, we can find $1 \leqq q \leqq p-5$ such that $q = p-l-1$. Hence, it follows from (A.9) together with the Cauchy–Schwarz inequality and (2.3) that

$$\big|D_n(l)\big| = O\left(\sum_{k=1}^{n} f_k^{l-1}\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2q}\sum_{k=1}^{n} f_k\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2p}\right)^{1/2} = o(\log s_n) \quad \text{a.s.}$$

b) If $p-1 \leqq l \leqq 2(p-1)$ with $l$ even, we can choose $1 \leqq q \leqq p$ such that $q = 2p-l-1$. Then, we deduce once again from (A.9) together with the Cauchy–Schwarz inequality and (2.3) that

$$\big|D_n(l)\big| = O\left(\sum_{k=1}^{n} f_k^{l-1}\sum_{k=1}^{n} f_k\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2q}\right)^{1/2} = o(\log s_n) \quad \text{a.s.}$$

**Case 3.** Assume that $3 \leqq l \leqq 2p-3$ with $l$ odd. Then, we can find $1 \leqq q \leqq p-2$ such that $2q = 2p-l-1$. Consequently, we immediately obtain from (A.9) and (2.3) that

$$\big|D_n(l)\big| = O\left(\sum_{k=1}^{n} f_k^{l/2}\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2q}\right) = o(\log s_n) \quad \text{a.s.}$$

Therefore, (A.8) clearly follows from the above three cases. Finally, we find from (A.2) together with (A.5) and (A.6) that $\mathcal{A}_n = O(\log s_n)$ a.s. Furthermore, we have the decomposition $f_n = a_n(p)v_n(p)$ where $a_n(p)$ is given by

$$a_n(p) = \frac{1-b_n^2}{1-b_n^{2p-1}} \quad \text{with} \quad b_n = \frac{\sqrt{s_{n-1}}}{\sqrt{s_n}}.$$

As $b_n$ tends to 1 a.s., we obtain by use of L'Hopital's rule that $a_n(p)$ converges to $2/(2p-1)$ a.s. Whence, as $\mathcal{A}_n = o(\log s_n)$ a.s., it ensures that a.s.

$$\sum_{k=1}^{n} f_k V_k = o(\log s_n) + o(T_n) \quad \text{with} \quad T_n = \sum_{k=1}^{n} v_k(p)|V_k|. \tag{A.11}$$

In addition, we obtain from the Hölder inequality

$$T_n \leqq \left(\sum_{k=1}^{n} v_k(p)\right)^{1/2p}\left(\sum_{k=1}^{n} v_k(p)\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2p}\right)^{1-1/2p}$$

However, by the convexity of the function $x^{p-1/2}$, we have for all $n \geqq 1$ and for any $p \geqq 2$, $2(2p-1)^{-1} \leqq a_n(p) \leqq 1$ which implies that $v_n(p) \leqq p f_n$ and

$$T_n \leqq p\left(\sum_{k=1}^{n} f_k\right)^{1/2p}\left(\sum_{k=1}^{n} f_k\left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2p}\right)^{1-1/2p}.$$

Finally, it follows from (2.3) that $T_n = O(\log s_n)$ a.s. which, by use of (A.11), leads to convergence (2.4) completing the proof of Theorem 3. $\square$

## Appendix B

Appendix B deals with the

Proof of Theorem 5. As in Appendix A, we shall only study convergence for odd moments (3.3) as convergence for even moments (3.2) was already established in [2]. We shall prove convergence (3.3) by induction on the power $p \geqq 1$ with a repeated use of decomposition (A.2). For $p = 1$, we already saw that $V_{n+1}^2 = O(\log s_n)$ a.s. In addition, as the explosion coefficient $f_n$ converges a.s. to $f$, $s_{n-1}/s_n$ tends a.s. to $1-f$ and $\log s_n$ is a.s. equivalent to $-n\log(1-f)$. Consequently, we obtain that $V_{n+1}^2 = O(n)$ which leads to $V_{n+1} = o(n)$ a.s. Moreover, $(\mathcal{W}_n)$ is a locally square integrable martingale with increasing process $(\langle\mathcal{W}\rangle_n)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{W}_{n+1} = \sigma^2 f \quad \text{a.s.}$$

Hence, according to the standard strong law of large numbers for martingales $\mathcal{W}_{n+1} = o(n)$ a.s. Therefore, it clearly follows from (A.2) that $\mathcal{A}_n = o(n)$ a.s. Furthermore, as $f_n \to f$ a.s., $v_n(1)$ converges a.s. to $1-\sqrt{1-f}$. Consequently, we obtain that a.s.

$$\sum_{k=1}^{n} V_k = o(n) + o(T_n) \quad \text{with} \quad T_n = \sum_{k=1}^{n} |V_k|. \tag{B.1}$$

Moreover, it follows from the Cauchy–Schwarz inequality together with convergence (3.2) for $p = 1$ that $T_n = O(n)$ a.s. Thus, (B.1) immediately implies (3.3) for $p = 1$. Now, let $p \geqq 2$ and assume that convergence (3.3) holds for any power $q$ with $1 \leqq q \leqq p-1$. We already saw in Appendix A that

$V_{n+1} = o(\log s_n)$ so that $V_{n+1} = o(n)$ a.s. In addition, it follows from Chow's lemma that

$$\mathcal{W}_{n+1} = o\Big(\sum_{k=1}^{n} f_k^p\Big) + O\Big(\sum_{k=1}^{n} f_k^{p-1/2}\big|\sigma_k(2p-1)\big|\Big) \quad \text{a.s.}$$

Hence, as $f_n \to f$ and $\sigma_n(2p-1)$ tends to zero a.s., we deduce that

$$\mathcal{W}_{n+1} = o(n) \qquad \text{a.s.} \tag{B.2}$$

Next, via the same reasoning as in Appendix A, we find that for any $1 \leqq l \leqq 2(p-1)$, $C_{n+1}(l) = o(n)$ a.s which leads to

$$\mathcal{B}_{n+1} = \sum_{l=2}^{2p-2} C_{2p-1}^{l} D_n(l) + o(n) \quad \text{a.s.} \tag{B.3}$$

It remains to study the asymptotic behavior of $D_n(l)$ in the three following cases.

**Case 1.** Let $l = 2$. It follows from the induction assumption that for any integer $1 \leqq q \leqq p-1$

$$\sum_{k=1}^{n} \Big(\frac{M_k}{\sqrt{s_{k-1}}}\Big)^{2q-1} = o(n) \qquad \text{a.s.} \tag{B.4}$$

Then, we infer from (B.4) with $q = p-1$ that

$$D_n(2) = \sigma^2 \sum_{k=1}^{n} f_k(1-f_k)^{p-3/2} \Big(\frac{M_k}{\sqrt{s_{k-1}}}\Big)^{2p-3} = o(n) \quad \text{a.s.}$$

**Case 2.** Assume that $4 \leqq l \leqq 2(p-1)$ with $l$ even. We split $D_n(l)$ into two terms,

$$D_n(l) = \sigma(l) \sum_{k=1}^{n} \varphi_k(l) + \sum_{k=1}^{n} \varphi_k(l)\big(\sigma_k(l) - \sigma(l)\big). \tag{B.5}$$

Moreover, we can find $1 \leqq q \leqq p-2$ such that such that $2q = 2p - l$. Hence, we deduce from (B.4) that

$$\sum_{k=1}^{n} \varphi_k(l) = \sum_{k=1}^{n} f_k^{p-q}(1-f_k)^{q-1/2} \Big(\frac{M_k}{\sqrt{s_{k-1}}}\Big)^{2q-1} = o(n) \quad \text{a.s.}$$

Furthermore, it follows from the Hölder inequality that

$$\sum_{k=1}^{n} \big|\,\varphi_k(l)\big| \leqq \sum_{k=1}^{n} \left(\frac{|M_k|}{\sqrt{s_{k-1}}}\right)^{2q-1} \leqq n^{\rho/p} \left( \sum_{k=1}^{n} \left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2p} \right)^{1-\rho/p}$$

with $\rho = p - q + 1/2$ which, via (3.2), ensures that

$$\sum_{k=1}^{n} \big|\,\varphi_k(l)\big| = O(n) \quad \text{a.s.}$$

Consequently, we obtain from (3.1) and (B.5) that

$$D_n(l) = o(n) + o\bigg( \sum_{k=1}^{n} \big|\,\varphi_k(l)\big| \bigg) = o(n) \quad \text{a.s.}$$

**Case 3.** Assume that $3 \leqq l \leqq 2p-3$ with $l$ odd. Then, we can find $1 \leqq q \leqq p-2$ such that $2q = 2p - l - 1$ and we directly obtain from (3.2) that

$$\sum_{k=1}^{n} \big|\,\varphi_k(l)\big| = O\left( \sum_{k=1}^{n} \left(\frac{M_k}{\sqrt{s_{k-1}}}\right)^{2q} \right) = O(n) \quad \text{a.s.}$$

Whence, as $\sigma_n(l) \to 0$ a.s., we infer that

$$D_n(l) = O(1) + o\bigg( \sum_{k=1}^{n} \big|\,\varphi_k(l)\big| \bigg) = o(n) \quad \text{a.s.}$$

According to the above three cases, we find that for any $2 \leqq l \leqq 2(p-1)$, $D_n(l) = o(n)$ a.s. and we immediately deduce from (B.3) that

$$\mathcal{B}_{n+1} = o(n) \quad \text{a.s.} \tag{B.6}$$

Consequently, it follows from the conjunction of (A.2), (B.2) and (B.6) that $\mathcal{A}_n = o(n)$ a.s. Finally, as $f_n \to f$ a.s., $v_n(p)$ converges a.s. to $1-(1-f)^{p-1/2}$ which ensures that

$$\sum_{k=1}^{n} V_k = o(n) \quad \text{a.s.}$$

completing the proof of Theorem 5. $\square$

## REFERENCES


[1] Berkes, I. and Csáki, E., A universal result in almost sure central limit theory, *Stochastic Process. Appl.* **94** (2001), 105–134. *MR* **2002j**:60033

[2] Bercu, B., On the convergence of moments in the almost sure central limit theorem for martingales with statistical applications, *Stochastic Process. Appl.* **111** (2004), 157–173. *MR* **2005f**:60075

[3] Billingsley, P., *Probability and measure,* third edition John Wiley & Sons, New York, 1995. *MR* **95k**:60001

[4] Brosamler, G. A., An almost everywhere central limit theorem, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), 561–574. *MR* **89i**:60045

[5] Chaabane, F., Version forte du théorème de la limite centrale fonctionnel pour les martingales, *C.R. Acad. Sci. Paris* **323** (1996), 195–198. *MR* **97d**:60057

[6] Chaabane, F. and Maaouia, F., Théorèmes limites avec poids pour les martingales vectorielles, *Esaim Prob. Stat.* **4** (2000), 137–189. *MR* **2002f**:60035

[7] Chaabane, F., Invariance principle with logarithm averaging for martingales, *Studia Math. Sci. Hungar.* **37** (2001), 21–52. *MR* **2002b**:60030

[8] Duflo, M., *Random Iterative Models,* Springer Verlag, Berlin, 1997. *MR* **98m**:62239

[9] Feller, W., *An introduction to probability theory and its applications,* vol. II, John Wiley, New York, 1966. *MR* **35**#1048

[10] Guttorp, P., *Statistical Inference for Branching Processes,* John Wiley, New York, 1991. *MR* **94m**:62214

[11] Ibragimov, I. A. and Lifshits, M. A., On the convergence of generalized moments in almost sure central limit theorem, *Statist. Probab. Lett.* **40** (1998), 343–351. *MR* **99m**:60032

[12] Ibragimov, I. A. and Lifshits, M. A., On almost sure limit theorems, *Theory Probab. Appl.* **44** (2000), 254–272. *MR* **2001g**:60066

[13] Lacey, M. T. and Phillip, W., A note on the almost sure central limit theorem, *Statist. Probab. Lett.* **9** (1990), 201–205. *MR* **91e**:60100

[14] Lai, T. L. and Wei, C. Z., Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems, *Ann. Statist.* **10** (1982), 154–166. *MR* **84c**:62091

[15] Lai, T. L. and Wei, C. Z., Asymptotic properties of general autoregressive models and strong consistency of least-squares estimates of their parameters, *J. Multivariate Analysis* **13** (1983), 1–23.

[16] Lifshits, M., Almost sure limit theorem for martingales, in: *Limit Theorems in Probability and Statistics II* (I. Berkes, E. Csáki, M. Csörgő, eds.) J. Bolyai Mathematical Society, Budapest (2002), 367–390. *MR* **2004c**:60090

[17] Lifshits, M., Lecture notes on almost sure limit theorems, *Publications IRMA* **54** (2001), Lille, 1–25.

[18] Maaouia, F. and Touati, A., Identification of multitype branching processes, *Ann. Statist.* **33** (2005), 2655–2694. *MR* 2253098

[19] Schatte, P., On strong versions of the almost sure central limit theorem, *Math Nachr.* **137** (1988), 249–256. *MR* **89i**:60070

[20] Schatte, P., On the central limit theorem with almost sure convergence, *Probab. Math. Statist.* **11** (1991), 237–246. *MR* **92k**:60050

[21] Wei, C. Z., Adaptive prediction by least squares predictors in stochastic regression models with applications to time series, *Ann. Statist.* **15** (1987), 1667–1682. *MR* **82e**:62123