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SPECTRAL PROPERTIES OF CHAOTIC PROCESSES

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We investigate the spectral asymptotic properties of the stationary dynamical system $\xi_t = \varphi(T^t(X_0))$. This process is given by the iterations of a piecewise expanding map T of the interval $[0,1]$, invariant for an ergodic probability μ. The initial state X_0 is distributed over $[0,1]$ according to μ and φ is a function taking values in $\mathbb{R}$. We establish a strong law of large numbers and a central limit theorem for the integrated periodogram as well as for Fourier transforms associated with $(\xi_t : t \in \mathbb{N})$. Several examples of expanding maps T are also provided.

Keywords: Integrated periodogram; Fourier transforms; strong law; central limit theorem; dynamical systems.

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1. Introduction

Over the last decade, the statistical properties of chaotic processes have been investigated in order to modalize complex systems [1, 2, 15].

More precisely, chaotic dynamical systems such as expanding maps of the interval have been suggested to capture the complexity of packet traffic [15] or to analyze the measurements of communication traffic from a wide variety of sources [2].

Chaos is the phenomenon by which low order nonlinear dynamical systems exhibit complex, seemingly random behavior. One can notice that trajectories of chaotic systems are very often fractal in nature, hence they can be used as convenient generators of fractal structures.

In this paper, we shall focus our attention on the strictly stationary dynamical system given, for all $t \in \mathbb{N}$, by

$$\xi_t = \varphi(T^t(X_0)) = \varphi(X_t), \tag{1}$$

where T is a piecewise expanding map [19] of the interval $[0, 1]$, invariant for an ergodic probability μ. The initial state X_0 is distributed over $[0, 1]$ according to μ and φ is a function from $[0, 1]$ to $\mathbb{R}$. Under suitable assumptions on T and φ, we shall establish a strong law of large numbers and a central limit theorem for the integrated periodogram associated with $(\xi_t : t \in \mathbb{N})$. We shall also prove similar results for Fourier transforms of $(\xi_t : t \in \mathbb{N})$. We will apply our results to several parametric Lasota–Yorke maps [19] which are piecewise expanding maps of the interval $[0, 1]$. It is well known [6, 8, 17, 20, 21] that such maps admit a unique absolutely continuous invariant measure. One can find very few papers dealing with the problem of the nonparametric estimation of the invariant density [7, 9, 25]. Our purpose is now to analyze the spectral asymptotic properties of such chaotic processes. We shall assume that the process $(\xi_t : t \in \mathbb{N})$ is zero mean and we denote by $(\gamma(t))$ its covariogram defined, for all $t \geq 0$, by

$$\gamma(t) = \mathbb{E}_\mu[\xi_0 \xi_t]. \tag{2}$$

The spectral density of $(\xi_t : t \in \mathbb{N})$ is given, for all λ in the torus $\mathbb{T} = [-\pi, \pi[$, by

$$f(\lambda) = \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \gamma(|t|) e^{-it\lambda}. \tag{3}$$

If the maps φ and T are both continuous and $(\gamma(t))$ goes to zero at a polynomial rate of order > 2, Lopes and Lopes [22] have proven the convergence in distribution sense of the empirical periodogram to the spectral density f. One can observe that the periodogram is evaluated on the discrete Fourier frequencies.

More recently, Chazottes, Collet and Schmitt [7] studied the convergence in $\mathbb{L}^2$ of the empirical spectral distribution for a wide class of dynamical systems, including piecewise expanding maps of the interval $[0, 1]$ satisfying Devroye's inequality.

Our approach is totally different. We first propose central limit theorem for the integrated periodogram of $(\xi_t : t \in \mathbb{N})$, in the spirit of the original work of Rosenblatt [26]. This result only holds under several conditions on the stationary process and on its covariogram. We propose projective criteria, expressed with projective coefficients in the style of Gordin, under which assumptions for the central limit theorem are satisfied. We next introduce the time reversal process $(Y_t : t \in \mathbb{N})$ associated with the underlying process $(X_t : t \in \mathbb{N})$. The process $(Y_t : t \in \mathbb{N})$ is a Markov chain [4]. One can check that $(X_0, X_1, \ldots, X_t)$ shares the same distribution as $(Y_t, Y_{t-1}, \ldots, Y_0)$. Via Markov arguments, we show that the projective criteria are fulfilled for our reversed process $(\varphi(Y_t) : t \in \mathbb{N})$. Finally, by the use of similar techniques, we deduce from [30] the asymptotic behavior of Fourier transforms of $(\xi_t : t \in \mathbb{N})$ defined, for a given real-valued function g and for all $\theta \in \mathbb{R}$, by

$$S_n(\theta) = \sum_{t=1}^{n} g(\xi_t) e^{it\theta}.$$

The paper is organized as follows. In Sec. 2, we give the central limit theorem for the integrated periodogram of $(\xi_t : t \in \mathbb{N})$. Section 3 is devoted to the time reversal Markov chain $(Y_t : t \in \mathbb{N})$ associated with $(X_t : t \in \mathbb{N})$.

Coefficients of dependence, very useful to measure the dependence structure of the underlying process $(X_t : t \in \mathbb{N})$, are given in Sec. 4. We also propose projective criteria under which the assumptions of the central limit theorem are satisfied. By use of $(Y_t : t \in \mathbb{N})$, we show that the projective criteria are fulfilled so that we get the central limit theorem for the integrated periodogram of $(\xi_t : t \in \mathbb{N})$. Several examples of expanding maps T whose reversed process satisfies the projective criteria are provided in Sec. 5. Section 6 is devoted to the asymptotic results concerning the Fourier transforms of $(\xi_t : t \in \mathbb{N})$ while Sec. 7 deals with the non-stationary case. Finally, all the technical proofs are postponed to Sec. 8.

2. Integrated Periodogram

We shall now define the integrated periodogram associated with $(\xi_t : t \in \mathbb{N})$ given by (1) and investigate its asymptotic properties. Assume in the sequel that $\mathbb{E}_\mu[\xi_0^4]$ is finite. The fourth cumulants of $(\xi_t : t \in \mathbb{N})$ are given, for all $(r, s, t) \in \mathbb{Z}^3$, by

$$\kappa(r,s,t) = \mathbb{E}_\mu[\xi_0\xi_r\xi_s\xi_t] - \mathbb{E}_\mu[\xi_0\xi_r]\mathbb{E}_\mu[\xi_s\xi_t] - \mathbb{E}_\mu[\xi_0\xi_s]\mathbb{E}_\mu[\xi_r\xi_t] - \mathbb{E}_\mu[\xi_0\xi_t]\mathbb{E}_\mu[\xi_r\xi_s].$$

By stationarity of the process $(\xi_t : t \in \mathbb{N})$, those cumulants may be defined over $\mathbb{Z}^3$. It is natural [26, Corollary 1, p. 59] to assume that

$$\gamma = \sum_{t\in\mathbb{N}} \gamma(t)^2 < +\infty \quad \text{and} \quad \kappa = \sum_{(r,s,t)\in\mathbb{Z}^3} |\kappa(r,s,t)| < +\infty. \tag{4}$$

The empirical periodogram associated with $(\xi_t : t \in \mathbb{N})$ is defined, for all $\lambda \in \mathbb{T}$, by

$$I_n(\lambda) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} \xi_t e^{-it\lambda}\right|^2. \tag{5}$$

Let $(\gamma_n(t))$ be the empirical covariances given, for all $0 \le t \le n-1$, by

$$\gamma_n(t) = \frac{1}{n}\sum_{k=1}^{n-t} \xi_k \xi_{t+k}$$

and $\gamma_n(t) = 0$ if $t \ge n$. One can easily see from (5) that, for all $\lambda \in \mathbb{T}$

$$I_n(\lambda) = \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \gamma_n(|t|)e^{-it\lambda}. \tag{6}$$

It is well known that $I_n(\lambda)$ is not a good estimator of $f(\lambda)$.

It is more appropriate to make use of the integrated periodogram

$$\mathcal{I}_n(g) = \int_{-\pi}^{\pi} g(\lambda) I_n(\lambda)\, d\lambda\,, \tag{7}$$

where g belongs to the set of functions

$$\mathcal{G} = \{g : \mathbb{T} \to \mathbb{R}, 2\pi\text{-periodic and continuous with } g \in \mathbb{L}^2(\mathbb{T})\}.$$

We shall prove the almost sure convergence of $\mathcal{I}_n(g)$ to

$$\mathcal{I}(g) = \int_{-\pi}^{\pi} g(\lambda)f(\lambda)\,d\lambda$$

together with a central limit theorem which makes use of the bispectral density function

$$f_4(\lambda,\mu,\nu) = \frac{1}{(2\pi)^3}\sum_{(r,s,t)\in\mathbb{Z}^3}\kappa(r,s,t)e^{-i(r\lambda+s\mu+t\nu)}\,.$$

Theorem 1. *For all $t \in \mathbb{N}$,*

$$\lim_{n\to\infty}\gamma_n(t) = \gamma(t) \quad a.s.$$

In addition, assume that (4) *holds and, for all $t \in \mathbb{N}$,*

$$\sum_{s=1}^{\infty}(\mathbb{E}[(\mathbb{E}[\xi_0\xi_s|\xi_{t+s}] - \gamma(t))^2])^{1/2} < \infty \tag{8}$$

and

$$\sigma^2(t) = \mathbb{E}[(\xi_0\xi_t - \gamma(t))^2] + 2\sum_{s=1}^{\infty}\mathbb{E}[(\xi_s\xi_{t+s} - \gamma(t))(\xi_0\xi_t - \gamma(t))] > 0. \tag{9}$$

Then, for all $t \in \mathbb{N}$,

$$\sqrt{n}(\gamma_n(t) - \gamma(t)) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\sigma^2(t)). \tag{10}$$

Furthermore, for all s, $t \in \mathbb{N}$, set

$$\sigma(s,t) = \mathbb{E}[(\xi_0\xi_s - \gamma(s))(\xi_0\xi_t - \gamma(t))] + 2\sum_{r=1}^{\infty}\mathbb{E}[(\xi_r\xi_{t+r} - \gamma(t))(\xi_0\xi_s - \gamma(s))].$$

Then, for all $d \geq 1$ and for arbitrary distinct integers $t_1, t_2, \ldots, t_d \in \mathbb{N}$, we have

$$\{\sqrt{n}(\gamma_n(t_i) - \gamma(t_i))\}_{1\leq i\leq d} \xrightarrow{\mathcal{L}} \mathcal{N}_d(0,\Gamma), \tag{11}$$

where Γ is the positive definite covariance matrix given by

$$\Gamma = (\sigma(t_i,t_j))_{1\leq i,j\leq d}.$$

Theorem 2. *For all $g \in \mathcal{G}$,*

$$\lim_{n\to\infty}\mathcal{I}_n(g) = \mathcal{I}(g) \quad a.s. \tag{12}$$

In addition, under assumptions (4), (8) *and* (9), *we have the finite dimensional convergence of $\{\sqrt{n}(\mathcal{I}_n(g) - \mathcal{I}(g)),\ g \in \mathcal{G}\}$ to the zero mean Gaussian process $\{Z(g),\ g \in \mathcal{G}\}$ with covariance given, for all g_1, $g_2 \in \mathcal{G}$ by*

$$\Gamma(g_1,g_2) = 4\pi\int_{-\pi}^{\pi} g_1(\lambda)g_2(\lambda)f^2(\lambda)d\lambda + 2\pi\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g_1(\lambda)g_2(\mu)f_4(\lambda,-\mu,\mu)d\lambda d\mu.$$

Proof. The proof of Theorems 1 and 2 are postponed to Sec. 8. □

3. The Associated Markov Chain

For the process $(\xi_t : t \in \mathbb{N})$, the randomness only enters when setting the initial state. The analogy with classical process is therefore clearer when considering the reversed process, for which randomness enters progressively at each step. Starting from this remark of Barbour *et al.* [4], we introduce the time reversal process $(Y_t : t \in \mathbb{N})$ associated with the underlying process $(X_t : t \in \mathbb{N})$ given by (1). First of all, let $\mathcal{L}$ be the operator from $\mathbb{L}^1([0,1])$ to $\mathbb{L}^1([0,1])$ given, for all functions $h \in \mathbb{L}^1([0,1])$ and $k \in \mathbb{L}^\infty([0,1])$, by the identity

$$\int_0^1 \mathcal{L}(h)(x)k(x)\,dx = \int_0^1 h(x)k(T(x))\,dx.$$

The operator $\mathcal{L}$ is called the Perron–Frobenius operator of T.

Assume that the probability distribution μ is absolutely continuous with respect to the Lebesgue measure. Denote by f_μ the density function associated with μ. Let $I^* \subset [0,1]$ be the support of μ and choose a version of f_μ such that $f_\mu > 0$ on I^* and $f_\mu = 0$ otherwise. One can observe that it is possible to choose $\mathcal{L}$ such that $\mathcal{L}(f_\mu h)(x) = \mathcal{L}(f_\mu h)(x)\mathbb{I}_{f_\mu(x)>0}$. Let K be the Markov kernel associated to T given, for all $x \in [0,1]$, by

$$K(h)(x) = \frac{\mathcal{L}(f_\mu h)(x)}{f_\mu(x)}\mathbb{I}_{f_\mu(x)>0} + \mu(h)\mathbb{I}_{f_\mu(x)=0}. \tag{13}$$

The time reversal process $(Y_t : t \in \mathbb{N})$ associated with $(X_t : t \in \mathbb{N})$ is a stationary Markov chain with invariant distribution μ and transition kernel K. It is easy to check [4] that $(X_0, X_1, \ldots, X_t)$ shares the same distribution as $(Y_t, Y_{t-1}, \ldots, Y_0)$. Hence, to prove Theorem 1, we study the asymptotic behavior of

$$\sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n-t}\varphi(Y_k)\varphi(Y_{t+k}) - \gamma(t)\right).$$

Consequently, it is necessary to go further in the study of the reversed process $(Y_t : t \in \mathbb{N})$. Denote by $BV([0,1])$ the set of functions

$$BV([0,1]) = \{h : [0,1] \to \mathbb{R} \text{ such that } h \in \mathbb{L}^1([0,1]), \|Dh\| < +\infty\},$$

where $\|Dh\| = |Dh|([0,1])$ stands for the total variation of the distributional derivative of h on $[0,1]$. Of course, if h is absolutely continuous, Dh is a function which coincides with the pointwise derivative h' of h. The set $BV([0,1])$ is a Banach space endowed with the norm $\|h\|_v = \|h\|_1 + \|Dh\|$. In many interesting cases, the spectral analysis of the Perron–Frobenius operator $\mathcal{L}$ in the Banach space $BV([0,1])$ can be achieved by using the Ionescu–Tulcea and Marinescu theorem [6,18]. Assume that 1 is a simple eigenvalue of $\mathcal{L}$ and that the rest of the spectrum is contained in a closed disk of radius strictly smaller than one. Then, one can find a unique T-invariant absolutely continuous probability distribution μ whose density function f_μ belongs to $BV([0,1])$, such that all the powers of the Kernel K can be decomposed for all $x \in [0,1]$, as

$$K^n(h)(x) = \frac{\Psi^n(f_\mu h)(x)}{f_\mu(x)} \mathbb{I}_{f_\mu(x)>0} + \mu(h)\mathbb{I}_{f_\mu(x)=0} \tag{14}$$

with $\Psi(f_\mu) = 0$ and

$$\|\Psi^n(h)\|_v \leq c\rho^n \|h\|_v \tag{15}$$

for some $0 < \rho < 1$ and $c > 0$. In addition, assume that

$$d = \left\| \frac{1}{f_\mu} \mathbb{I}_{f_\mu>0} \right\|_v < \infty. \tag{16}$$

It was proven by Dedecker and Prieur [10] that, for any functions $h \in BV([0,1])$ and $k \in \mathbb{L}^1([0,1])$,

$$|\mathrm{Cov}(h(X_0), k(X_n))| \leq a_n \|k(X_n)\|_1 \|h\|_v, \tag{17}$$

where $a_n = \alpha\rho^n$ with $\alpha = 2cd(\|Df_\mu\| + 1)$.

We refer to the paper of Broise [6] for examples of dynamics T satisfying assumption (15) and to the paper of Morita [24] for sufficient conditions implying assumption (16). Two examples of piecewise expanding maps are detailed in Sec. 5.

4. Coefficients of Dependence

In order to check that the assumptions of Theorem 1 are satisfied, we introduce coefficients of dependence which allow us to measure the dependence structure of the reversed process $(\varphi(Y_t) : t \in \mathbb{N})$. For dynamical systems, and in particular for piecewise expanding maps of the interval, such coefficients of dependence are closely related to covariance inequalities.

Definition 1. Let (Z_t) be a stationary sequence of zero mean real-valued random variables. Let $\mathbb{F} = (\mathcal{F}_t)$ be the filtration given by $\mathcal{F}_t = \sigma(Z_0, Z_1, \ldots, Z_t)$ and denote by $\mathbb{E}_t$ the conditional expectation with respect to $\mathcal{F}_t$. For any integers $0 \leq i < j$ and $k \geq 0$, let $\Gamma_{i,j,k}$ be the set of multiintegers $(t_1, t_2, \ldots, t_j)$ such that $0 \leq t_1 \leq \cdots \leq t_i$ and $t_i + k \leq t_{i+1} \leq \cdots \leq t_j$. For $m \in \{1, 2\}$, set

$$\theta_{i,j,m}(k) = \sup_{(t_1,\ldots,t_j)\in\Gamma_{i,j,k}} \|Z_{t_1} \cdots Z_{t_i} \mathbb{E}_{t_i}[Z_{t_{i+1}} \cdots Z_{t_j} - \mathbb{E}[Z_{t_{i+1}} \cdots Z_{t_j}]]\|_m. \tag{18}$$

In the case where $m = 1$, these coefficients have been first introduced [11] to derive Esseen's mean central limit theorems for dependent sequences.

We now give projective criteria using these dependence coefficients under which the central limit theorem for the integrated periodogram of a stationary process with mean zero holds.

Theorem 3. *Assume that the reversed process* $(\varphi(Y_t) : t \in \mathbb{N})$ *associated to the process* $(\xi_t : t \in \mathbb{N})$ *given by* (1) *satisfies the following three conditions* (a), (b) *and* (c). *Then*, (4) *and* (8) *hold true for* $(\xi_t : t \in \mathbb{N})$.

(a) $\sum_{k=0}^{\infty} \theta_{1,2,1}(k) < +\infty$,
(b) *For any* $1 \le i < j \le 4$,

$$\sum_{k=0}^{\infty} k^{j-2} \theta_{i,j,1}(k) < +\infty,$$

(c) $\sum_{k=1}^{\infty} \theta_{0,2,2}(k) < +\infty$.

Proof. The proof of Theorem 3 is postponed to Sec. 8. □

Theorem 4. *Assume that* (15) *and* (16) *hold. Moreover, suppose that* φ *is in* $BV([0,1])$. *Then, assumptions* (a), (b) *and* (c) *of Theorem* 3 *are satisfied.*

Proof. The proof is given in Sec. 8. □

Remark. Under assumptions (15) and (16), we know from Theorems 3 and 4 that (4) and (8) hold true for $(\xi_t : t \in \mathbb{N})$. But we did not say anything about assumption (9) of Theorem 1. The case

$$\sigma^2(t) = \mathbb{E}[(\xi_0 \xi_t - \gamma(t))^2] + 2 \sum_{s=1}^{\infty} \mathbb{E}[(\xi_s \xi_{t+s} - \gamma(t))(\xi_0 \xi_t - \gamma(t))] = 0$$

is a degenerate case. The result of Theorem 1 still holds but the limit is a Gaussian law with variance zero. Let us give a simple situation under which we can prove that we are not in the degenerate case. We know from [26, Chap. III] that the limiting variance in Theorem 1 satisfies

$$\sigma^2(t) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n} \sum_{l=0}^{n} \operatorname{Cov}(\xi_k \xi_{k+t}, \xi_l \xi_{l+t}). \tag{19}$$

Let us endow $BV([0,1])$ with the scalar product defined, for all h, k in $BV([0,1])$, by

$$\langle h, k \rangle_\mu = \int_0^1 h(x) k(x) f_\mu(x) dx.$$

For all $x \in [0,1]$ and $t \ge 0$, set $h_t(x) = \varphi(x)\varphi(T^t(x)) - \gamma(t)$.

Relation (19) can be rewritten as

$$\sigma^2(t) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n} \sum_{l=0}^{n} \langle h_t(T^k), h_t(T^l) \rangle_\mu = \sum_{k \in \mathbb{Z}} \langle h_t(T^k), h_t \rangle_\mu .$$

Hence, it follows from Proposition 7.1 of [6] that, for a piecewise expanding map T, if the associated partition is countably infinite, and if assumption (16) is satisfied, then the limit $\sigma^2(t)$ is strictly positive as soon as the function φ is not a constant. The proof relies on Ionescu–Tulcea and Marinescu Theorem on the spectral decomposition of the Perron–Frobenius operator $\mathcal{L}$ associated to T.

5. Piecewise Expanding Maps

Several examples of expanding maps T satisfying conditions (15) and (16) are given in [6, p. 11]. We shall now focus our attention on two specific transformations. On the one hand, for some integer $\beta \geq 2$ and for all $x \in [0,1]$, let

$$T_\beta(x) = \beta x - [\beta x]. \tag{20}$$

This map is commonly called β-transformation. The invariant probability μ is the Lebesgue measure on $[0,1]$. A straightforward calculation leads to

$$\mathbb{E}[X_0] = \frac{1}{2} \quad \text{and} \quad \mathrm{Var}(X_0) = \frac{1}{12}.$$

Consequently, for all $x \in [0,1]$, we choose $\varphi(x) = x - 1/2$. It is not hard to see that for all $t \geq 0$

$$\gamma(t) = \frac{1}{12\beta^t}.$$

Hence, if we set

$$\sigma^2 = \frac{\beta^2 - 1}{12},$$

we deduce via (3) that the spectral density associated with (20) is given, for all $\lambda \in \mathbb{T}$, by

$$f(\lambda) = \frac{\sigma^2}{2\pi(1 + \beta^2 - 2\beta\cos(\lambda))}.$$

We can estimate the unknown parameter β by the Yule–Walker estimator

$$\widehat{\beta}_n = \frac{\gamma_n(0)}{\gamma_n(1)} = \frac{\sum_{k=1}^{n} \xi_k^2}{\sum_{k=1}^{n-1} \xi_k \xi_{k+1}}.$$

It immediately follows from Theorem 1 that $\widehat{\beta}_n \to \beta$ a.s. Furthermore, we have the decomposition

$$\sqrt{n}(\widehat{\beta}_n - \beta) = \frac{\sqrt{n}}{\gamma_n(1)}\left(\gamma_n(0) - \gamma(0) - \frac{\gamma(0)}{\gamma(1)}(\gamma_n(1) - \gamma(1))\right).$$

Consequently, we infer from (11) together with Slutsky's lemma that

$$\sqrt{n}(\widehat{\beta}_n - \beta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau^2),$$

where

$$\tau^2 = (12\beta)^2(\sigma^2(0) + \beta^2\sigma^2(1) - 2\beta\sigma(0,1)).$$

On the other hand, consider the transformation given, for some $0 < a < 1$ and for all $x \in [0,1]$, by

$$T_a(x) = \begin{cases} \dfrac{x}{a} & \text{if } 0 \le x < a, \\ \dfrac{1-x}{1-a} & \text{if } a \le x < 1. \end{cases} \tag{21}$$

As before, the invariant probability μ is the Lebesgue measure on $[0,1]$ so that we also take, for all $x \in [0,1]$, $\varphi(x) = x - 1/2$. One can easily see [23, p. 26] that for all $t \ge 0$

$$\gamma(t) = \frac{1}{12}(2a-1)^t.$$

If

$$\sigma^2 = \frac{a(1-a)}{6},$$

the spectral density associated with (21) is given, for all $\lambda \in \mathbb{T}$, by

$$f(\lambda) = \frac{\sigma^2}{2\pi(1 - 2a(1-a) + (2a-1)\cos(\lambda))}.$$

We can estimate the unknown parameter a by the Yule–Walker estimator

$$\widehat{a}_n = \frac{1}{2}\left(\frac{\gamma_n(1)}{\gamma_n(0)} + 1\right) = \frac{1}{2}\left(\frac{\sum_{k=1}^{n-1} \xi_k \xi_{k+1}}{\sum_{k=1}^{n} \xi_k^2} + 1\right).$$

We immediately deduce from Theorem 1 that $\widehat{a}_n \to a$ a.s. In addition, one can observe that

$$\sqrt{n}(\widehat{a}_n - a) = \frac{\sqrt{n}}{2\gamma_n(0)}\left(\gamma_n(1) - \gamma(1) - \frac{\gamma(1)}{\gamma(0)}(\gamma_n(0) - \gamma(0))\right).$$

Hence, we derive from (11) together with Slutsky's lemma that

$$\sqrt{n}(\widehat{a}_n - a) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau^2),$$

where

$$\tau^2 = 36(\sigma^2(1) + (2a-1)^2\sigma^2(0) - 2(2a-1)\sigma(0,1)).$$

6. Fourier Transforms

In this section, we investigate the asymptotic behavior of Fourier transforms of $(\xi_t : t \in \mathbb{N})$ given, for all $g \in BV([0,1])$ and all $\theta \in \mathbb{R}$, by

$$S_n(\theta) = \sum_{t=1}^{n} g(\xi_t) e^{it\theta}.$$

We shall assume that the function φ belongs to $BV([0,1])$ and that the expanding map T satisfies (15) and (16).

In addition, we suppose that $\mathbb{E}_\mu[g(\xi_0)] = 0$ and $\mathbb{E}_\mu[g^2(\xi_0)]$ is finite. It is obvious to realize that $S_n(\theta)$ shares the same distribution as

$$\Sigma_n(\theta) = \sum_{t=1}^{n} g(\varphi(Y_t))e^{it\theta},$$

where $(Y_t : t \in \mathbb{N})$ is the associated Markov chain of the underlying process $(X_t : t \in \mathbb{N})$. Consequently, we can deduce from [30], which deals with Fourier transforms of stationary and ergodic Markov chains, the asymptotic behavior of $S_n(\theta)$.

Corollary 1. *Assume that* (15) *and* (16) *are satisfied. Assume, moreover, that g is in $BV([0,1])$ and satisfies $\mathbb{E}_\mu[g(\xi_0)] = 0$ and $\mathbb{E}_\mu[g^2(\xi_0)]$ is finite. Then, for almost all $\theta \in \mathbb{R}$, there exists $0 \le \sigma(\theta) < \infty$ such that*

$$\frac{S_n(\theta)}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2(\theta)\mathrm{Id}_2). \tag{22}$$

In addition, for almost all pairs $(\theta, \lambda) \in \mathbb{R}^2$, $S_n(\theta)/\sqrt{n}$ and $S_n(\lambda)/\sqrt{n}$ are asymptotically independent. Finally, for almost all $\theta \in \mathbb{R}$, the spectral density of the process $(g(\xi_t) : t \in \mathbb{N})$ is given by $f_g(\theta) = \sigma^2(\theta)/\pi$.

Corollary 2. *Assume that* (15) *and* (16) *are satisfied. Assume, moreover, that g is in $BV([0,1])$ and satisfies $\mathbb{E}_\mu[g(\xi_0)] = 0$ and $\mathbb{E}_\mu[g^2(\xi_0)]$ is finite. Then, for all $\theta \in [0, 2\pi[$ there exists $h_\theta(\xi_0, \xi_1) \in \mathbb{L}^2([0,1])$ such that*

$$\sum_{t=1}^{n} e^{it\theta}(\mathbb{E}[g(\xi_t)|\xi_1] - \mathbb{E}[g(\xi_t)|\xi_0]) \tag{23}$$

converges in $\mathbb{L}^2([0,1])$ to $h_\theta(\xi_0, \xi_1)$, as n goes to infinity. In addition, we also have $\mathbb{E}[|\mathbb{E}[S_n(\theta)|\xi_0]|^2] = o(n)$. Moreover,

(i) *if $\theta \neq 0$, π,*

$$\frac{S_n(\theta)}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2(\theta)\mathrm{Id}_2), \tag{24}$$

where $\sigma^2(\theta) = \mathbb{E}[h_\theta^2(\xi_0, \xi_1)]/2$;

(ii) *if $\theta = 0$ or π,*

$$\frac{S_n(\theta)}{\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau^2(\theta))$$

with $\tau^2(\theta) = 2\sigma^2(\theta)$.

Proof. The proofs of Corollaries 1 and 2 are postponed to Sec. 8. □

Remark. In the last case $\theta = 0$ or π, the result is the central limit theorem stated for example in [6] with limiting variance given by

$$\tau^2(\theta) = \sum_{k=0}^{\infty} \langle g(\varphi(T^k)), g(\varphi) \rangle_\mu.$$

7. On the Non-Stationary Case

We shall now focus our attention on the asymptotic behavior of the non-stationary process $(\xi'_t : t \in \mathbb{N})$ given, for all $t \in \mathbb{N}$, by

$$\xi'_t = \varphi(T^t(X'_0)) = \varphi(X'_t), \tag{25}$$

where T is a piecewise expanding map of the interval $[0,1]$ and $\varphi \in BV([0,1])$. The initial state X'_0 is not distributed over $[0,1]$ according to μ but X'_0 has a probability density function $p \in BV([0,1])$. From a statistical point of view, it is indeed interesting to carry out the asymptotic analysis of functionals which do not depend on the unknown invariant density function f_μ. In many situations, it is possible to prove the convergence of the non-stationary case to the stationary one. To be more precise, let us recall the following inequality [8, 20, 29] on the Perron–Frobenius operator $\mathcal{L}$ in the case where the dynamical system is generated by a Lasota–Yorke map T. One can find some $0 < \delta < 1$ and $c > 0$ such that, for any $n \geq 0$

$$\|\mathcal{L}^n p - f_\mu\|_\infty \leq c\delta^n. \tag{26}$$

This result allows us to start from any initial state X'_0 with probability density function $p \in BV([0,1])$. Let us explain in details what happens in the case of the periodogram. We know from Sec. 2 that for any $t \geq 0$

$$S_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n-t} \xi_k \xi_{k+t} - \gamma(t)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2(t)). \tag{27}$$

We want to show that

$$S'_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n-t} \xi'_k \xi'_{k+t} - \gamma(t)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2(t)). \tag{28}$$

Denote by $\mathcal{H}$ the set of bounded differentiable functions $h : \mathbb{R} \to \mathbb{R}$ with continuous and bounded derivative. The set $\mathcal{H}$ is dense in $\mathcal{C}_b(\mathbb{R})$. Hence, it is only necessary to prove that, for any $h \in \mathcal{H}$ and $t \geq 0$, the difference

$$\Delta_n(h,t) = \mathbb{E}[h(S_n(t)) - h(S'_n(t))]$$

goes to zero as n tends to infinity. Let $h \in \mathcal{H}$ and denote $\theta = \|h\|_\infty$ and $\vartheta = \|h'\|_\infty$. For any $1 \leq l \leq n - t$, set

$$S_{n,l}(t) = \frac{1}{\sqrt{n}}\sum_{k=1}^{l} \xi_k \xi_{k+t} \quad \text{and} \quad S'_{n,l}(t) = \frac{1}{\sqrt{n}}\sum_{k=1}^{l} \xi'_k \xi'_{k+t}.$$

As $\varphi \in BV([0,1])$, φ is bounded by some constant $M > 0$. Hence, both $|S_{n,l}(t)|$ and $|S'_{n,l}(t)|$ are bounded by $M^2 l/\sqrt{n}$.

Furthermore, if $R_{n,l}(t) = S_n(t) - S_{n,l}(t)$ and $R'_{n,l}(t) = S'_n(t) - S'_{n,l}(t)$, we can deduce from the mean-valued theorem that

$$|h(S_n(t)) - h(R_{n,l}(t))| \leq \frac{\vartheta M^2 l}{\sqrt{n}} \quad \text{and} \quad |h(S'_n(t)) - h(R'_{n,l}(t))| \leq \frac{\vartheta M^2 l}{\sqrt{n}}.$$

Consequently, for any $1 \le l \le n-t$, we obtain that

$$|\Delta_n(h,t)| \le |\mathbb{E}[h(R_{n,l}(t)) - h(R'_{n,l}(t))]| + \frac{2\vartheta M^2 l}{\sqrt{n}}.$$

However,

$$\mathbb{E}[h(R_{n,l}(t)) - h(R'_{n,l}(t))] = \int_0^1 h(Z_{n,l}(x,t))(f_\mu(x) - \mathcal{L}^l p(x))\, dx,$$

where

$$Z_{n,l}(x,t) = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n-t-l} \varphi(T^k(x))\varphi(T^{k+t}(x)) - \gamma(t)\right).$$

Therefore, we infer from (26) that, for any $1 \le l \le n-t$

$$|\Delta_n(h,t)| \le \theta c \delta^l + \frac{2\vartheta M^2 l}{\sqrt{n}}.$$

Finally, as $0 < \delta < 1$, we can conclude that, for l large enough, $\Delta_n(h,t)$ goes to zero as n tends to infinity. $\square$

8. Proofs

8.1. *Proof of Theorems* 1 *and* 2

For all $t \in \mathbb{N}$, we immediately deduce from the ergodic theorem that $\gamma_n(t)$ converges almost surely to $\gamma(t)$. Then, the strong law (12) clearly follows from Levy's theorem [3, Theorem 2.2, p. 106]. The finite dimensional central limit of Theorem 1 is a direct application of [26, Theorem 3, p. 58] applied to the reversed process $(\varphi(Y_t) : t \in \mathbb{N})$ which is an ergodic strictly stationary sequence with mean zero. Assume that the process $(\xi_t : t \in \mathbb{N})$ satisfies assumptions (4), (8) and (9). For all $t \in \mathbb{N}$, $(X_0, X_1, \dots, X_t)$ shares the same distribution as $(Y_t, Y_{t-1}, \dots, Y_0)$. Consequently, assumptions (4) and (9) hold for the process $(\xi_t : t \in \mathbb{N})$, they also hold for the reversed process $(\varphi(Y_t) : t \in \mathbb{N})$. Moreover, assumption (8) yields

$$\sum_{s=1}^{\infty} (\mathbb{E}[(\mathbb{E}[\varphi(Y_s)\varphi(Y_{t+s})|\varphi(Y_0)] - \gamma(t))^2])^{1/2} < +\infty\,.$$

Hence, assumptions in [26, Theorem 3, p. 58] are satisfied for the reversed process $(\varphi(Y_t) : t \in \mathbb{N})$, which implies the finite dimensional central limit theorem for $(\varphi(Y_t) : t \in \mathbb{N})$ and therefore for $(\xi_t : t \in \mathbb{N})$. Finally, we complete the proof of Theorem 2 by the use of [26, Corollary 2, p. 61]. $\square$

8.2. *Proof of Theorem* 3

First of all, we shall prove that the projective criterion (a) implies that γ is finite.

For all $t \ge 0$, we have

$$|\gamma(t)| = |\mathbb{E}[\xi_0 \xi_t]| = |\mathbb{E}[\varphi(Y_0)\varphi(Y_t)]|\,.$$

Moreover,

$$|\mathbb{E}[\varphi(Y_0)\varphi(Y_t)]| = |\mathbb{E}[\varphi(Y_0)\mathbb{E}_0[\varphi(Y_t)]]| \le \|\varphi(Y_0)\mathbb{E}_0[\varphi(Y_t)]\|_1 \le \theta_{1,2,1}(t).$$

Therefore, if (a) holds,

$$\sum_{t=0}^{\infty} |\gamma(t)| \le \sum_{t=0}^{\infty} \theta_{1,2,1}(t) < +\infty$$

which immediately implies that γ is finite. Next, let us prove that (a) and (b) lead to κ finite. To prove that the sum κ of the fourth cumulants of the process $(\xi_t : t \in \mathbb{N})$ is finite, we prove that the sum τ of the fourth cumulants of the reversed process $(\varphi(Y_t) : t \in \mathbb{N})$ is finite. For this, we shall proceed by induction as in [12] for strong mixing processes. It is necessary to introduce some notations. For $n \ge 1$ and n real-valued random variables $A_1, \dots, A_n$, define

$$\mathrm{cum}(A_1, \dots, A_n) = \sum (-1)^{k-1}(k-1)!\mathbb{E}[\Pi_{i\in\nu_1} A_i] \cdots \mathbb{E}[\Pi_{i\in\nu_k} A_i], \tag{29}$$

where $\nu_1, \dots, \nu_k$ is a partition of $(1, 2, \dots, n)$ and one sums over all these partitions. Moreover, let ν be a subset of $\{1, \dots, n\}$ and define $C_\nu = \mathrm{cum}(A_i, i \in \nu)$.

We have from [26, Chap. II] that

$$\mathbb{E}[A_1 \cdots A_n] = \sum C_{\nu_1} \cdots C_{\nu_k}, \tag{30}$$

where one sums over all partitions $\nu_1, \dots, \nu_k$ of $(1, 2, \dots, n)$. As $(\varphi(Y_t) : t \in \mathbb{N})$ is centered, we get that for all $(r, s, t) \in \mathbb{N}^3$, $\tau(r, s, t) = \mathrm{cum}(\varphi(Y_0), \varphi(Y_r), \varphi(Y_s), \varphi(Y_t))$. For $2 \le p \le 4$, let

$$C_p = \sum_{0=t_1\le\cdots\le t_p} |c(t_1, \cdots, t_p)|,$$

where $c(t_1, \dots, t_p) = \mathrm{cum}(\varphi(Y_{t_1}), \dots, \varphi(Y_{t_p}))$.

We already saw that

$$C_2 = \sum_{t=0}^{\infty} |\gamma(t)| < +\infty.$$

Our purpose is to show by induction that, under (a) and (b), C_4 is finite.

For $p \ge 3$ and $0 = t_1 \le \cdots \le t_p$, let $r = t_{m+1} - t_m$ where

$$m = \inf\{1 \le m < p / t_{m+1} - t_m = \max(t_2 - t_1, \dots, t_p - t_{p-1})\}.$$

One can observe that $|c(t_1, \dots, t_p)|$ is bounded by

$$|\mathbb{E}[\varphi(Y_{t_1}) \cdots \varphi(Y_{t_p})] - \mathbb{E}[\varphi(Y_{t_1}) \cdots \varphi(Y_{t_m})]\mathbb{E}[\varphi(Y_{t_{m+1}}) \cdots \varphi(Y_{t_p})]| + R \tag{31}$$

with

$$R \le \sum_{k\neq 1} \frac{1}{k} |\mathbb{E}[\Pi_{i\in\nu_1}\varphi(Y_i)]| \cdots |\mathbb{E}[\Pi_{i\in\nu_k}\varphi(Y_i)]|,$$

where $\nu_1, \ldots, \nu_k$ is a partition of $(t_1, \ldots, t_p)$ with $k \neq 1$. By use of (30), one can find some constant $M(p) > 0$ such that

$$R \leq M(p) \sum_{k \neq 1} \frac{1}{k} \max |C_{\nu_1} \cdots C_{\nu_k}|, \tag{32}$$

where the maximum is taken over all partitions $\nu_1, \ldots, \nu_k$ of $(t_1, \ldots, t_p)$ with $k \neq 1$.

It follows from (18) that

$$|\mathbb{E}[\varphi(Y_{t_1}) \cdots \varphi(Y_{t_p})] - \mathbb{E}[\varphi(Y_{t_1}) \cdots \varphi(Y_{t_m})]\mathbb{E}[\varphi(Y_{t_{m+1}}) \cdots \varphi(Y_{t_p})]| \leq \theta_{m,p,1}(r). \tag{33}$$

Hence, we can deduce from (31) to (33) that

$$C_p \leq \sum_{r=0}^{\infty} r^{p-2} \theta_{m,p,1}(r) + M(p) \sum_{k \neq 1} \frac{1}{k} \sum_{p_1 + \cdots + p_k = p} C_{p_1} \cdots C_{p_k}.$$

The first right-hand term is finite because of (b). In addition, the second right-hand term is also finite by induction as all $p_i < p$. We can conclude that C_4 and τ are finite.

It remains to prove that (c) implies condition (8). This implication clearly follows from the fact that for all s, $t \geq 0$

$$\mathbb{E}[\xi_0 \xi_s | \xi_{t+s}] = \mathbb{E}[\varphi(Y_s)\varphi(Y_{t+s})|\varphi(Y_0)]$$

and

$$(\mathbb{E}[(\mathbb{E}[\varphi(Y_s)\varphi(Y_{t+s})|\varphi(Y_0)] - \gamma(t))^2])^{1/2} \leq \theta_{0,2,2}(s)$$

which completes the proof of Theorem 3. □

8.3. *Proof of Theorem* 4

In order to prove Theorem 4, assume that $\varphi \in BV([0,1])$ and let $C = \|D\varphi\|$. Denote by $BV_1([0,1])$ the subset of all functions $h \in BV([0,1])$ whose bounded variation norm is smaller than 1, that is $\|Dh\| \leq 1$. As φ is bounded by some constant $M > 0$, it follows from (18) that for any integers $0 \leq i < j$, $k \geq 0$ and for any $t = (t_1, t_2, \ldots, t_j) \in \Gamma = \Gamma_{i,j,k}$ and $m \in \{1, 2\}$,

$$\begin{aligned}
\theta_{i,j,m}(k) &= \sup_{t \in \Gamma} \|\varphi(Y_{t_1}) \cdots \varphi(Y_{t_i})\mathbb{E}_{t_i}[\varphi(Y_{t_{i+1}}) \cdots \varphi(Y_{t_j}) - \mathbb{E}[\varphi(Y_{t_{i+1}}) \cdots \varphi(Y_{t_j})]]\|_m \\
&\leq \sup_{t \in \Gamma} \|\varphi(Y_{t_1}) \cdots \varphi(Y_{t_i})\mathbb{E}_{t_i}[\varphi(Y_{t_{i+1}}) \cdots \varphi(Y_{t_j}) - \mathbb{E}[\varphi(Y_{t_{i+1}}) \cdots \varphi(Y_{t_j})]]\|_\infty \\
&\leq M^i \sup_{t \in \Gamma} \|\mathbb{E}_{t_i}[\varphi(Y_{t_{i+1}}) \cdots \varphi(Y_{t_j}) - \mathbb{E}[\varphi(Y_{t_{i+1}}) \cdots \varphi(Y_{t_j})]]\|_\infty \\
&\leq M^i C^{j-i} \Phi_{j-i}(k),
\end{aligned} \tag{34}$$

with $\mathcal{F}_{t_i} = \sigma(\varphi(Y_0), \ldots, \varphi(Y_{t_i})) \subset \mathcal{G}_{t_i} = \sigma(Y_0, \ldots, Y_{t_i})$,

$$\Phi_{j-i}(k) = \max_{1 \le l \le j-i} \sup_{t_i + k \le t_{i+1} \le \cdots \le t_{i+l}} \phi(\mathcal{G}_{t_i}, Y_{t_{i+1}}, \ldots, Y_{t_{i+l}}),$$

$$\phi(\mathcal{G}_{t_i}, Y_{t_{i+1}}, \ldots, Y_{t_{i+l}}) = \|\Delta_{il}\|_\infty,$$

$$\Delta_{il} = \sup \left| \mathbb{E}_{\mathcal{G}_{t_i}} \left[\prod_{s=1}^{l} (\varphi_s(Y_{t_{i+s}}) - \mathbb{E}[\varphi_s(Y_{t_{i+s}})]) \right] - \mathbb{E}\left[\prod_{s=1}^{l} (\varphi_s(Y_{t_{i+s}}) - \mathbb{E}[\varphi_s(Y_{t_{i+s}})]) \right] \right|,$$

where one takes the supremum over all functions $\varphi_1, \ldots, \varphi_l \in BV_1([0,1])$ and where $\mathbb{E}_{\mathcal{G}_{t_i}}$ is the conditional expectation with respect to $\mathcal{G}_{t_i}$. Under assumptions (15) and (16), we have for any $h \in BV([0,1])$,

$$\begin{aligned} \|DK^n(h)\| = \|DK^n(h - h(0))\| &\le 2d\|\Psi^n(f_\mu(h - h(0)))\|_v, \\ &\le 4\alpha\rho^n\|Dh\|, \end{aligned}$$

where $\alpha = 2cd(\|Df_\mu\| + 1)$. Hence, we find using exactly the same lines as [10, Lemma 1] that for all $k \ge 0$

$$\Phi_3(k) \le 2\alpha(1 + \beta + \beta^2)\rho^k \tag{35}$$

with $\beta = 4\alpha$. For any integers $0 \le i < j$, one can deduce from the definition of Φ_{j-i} that, for all $k \in \mathbb{N}$,

$$\Phi_1(k) \le \Phi_2(k) \le \Phi_3(k)\,.$$

Therefore, under assumptions (15) and (16) of Theorem 4, it clearly follows from (34) and (35) that for $0 \le i < j \le 4$, $m \in \{1, 2\}$, the coefficients $\theta_{i,j,m}(k)$ decrease exponentially fast to zero as k tends to infinity. Hence the three conditions of Theorem 3 are satisfied, which concludes the proof of Theorem 4. □

8.4. *Proof of Corollary* 1

Corollary 1 immediately follows from Theorem 1 and Proposition 2 of [30]. More precisely, we have to check that condition (2) in [30] is satisfied, so that

$$\sum_{t=1}^{\infty} \frac{1}{t} \mathbb{E}[(\mathbb{E}[g(\varphi(Y_t))|\varphi(Y_0)])^2] < \infty.$$

For all $t \ge 0$, we have

$$\mathbb{E}[(\mathbb{E}[g(\varphi(Y_t))|\varphi(Y_0)])^2] \le \theta_{0,1,2}^2(t).$$

In addition, under (15) and (16), $\theta_{0,1,2}(t)$ goes exponentially fast to zero as t tends to infinity. Hence, condition (2) in [30] is satisfied. □

8.5. *Proof of Corollary* 2

For $\theta = 0, \pi$, Corollary 2 follows from the classical central limit theorem for partial sums. It is stated for example in [6]. For $\theta \neq 0, \pi$, Corollary 2 can be proven by use of Theorem 2 of [30].

More precisely, we have to check that condition (7) in [30] is satisfied, so that

$$\sum_{t=1}^{\infty} \|\mathbb{E}[g(\varphi(Y_t))|\varphi(Y_1)] - \mathbb{E}[g(\varphi(Y_t))|\varphi(Y_0)]\|_2 < +\infty.$$

By the triangular inequality, $\|\mathbb{E}[g(\varphi(Y_t))|\varphi(Y_1)] - \mathbb{E}[g(\varphi(Y_t))|\varphi(Y_0)]\|_2$ is bounded by

$$\|\mathbb{E}[g(\varphi(Y_t))|\varphi(Y_1)]\|_2 + \|\mathbb{E}[g(\varphi(Y_t))|\varphi(Y_0)]\|_2 \,. \tag{36}$$

Both right-hand terms in inequality (36) decrease exponentially fast to zero as t tends to infinity, which concludes the proof of Corollary 2. □

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