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On the center of mass of the elephant random walk

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Abstract

Our goal is to investigate the asymptotic behavior of the center of mass of the elephant random walk, which is a discrete-time random walk on integers with a complete memory of its whole history. In the diffusive and critical regimes, we establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for the center of mass of the elephant random walk. The asymptotic normality, properly normalized, is also provided. Finally, we prove a strong limit theorem for the center of mass in the superdiffusive regime. All our analysis relies on asymptotic results for multi-dimensional martingales.

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Keywords: Elephant random walk; Center of mass; Multi-dimensional martingales; Almost sure convergence; Asymptotic normality

1. Introduction

Let (S_n) be a standard random walk in $\mathbb{R}^d$. The center of mass G_n of S_n is defined by

$$G_n = \frac{1}{n}\sum_{k=1}^{n} S_k. \tag{1.1}$$

The question of the asymptotic behavior of G_n was first raised by Paul Erdös. Very recently, Lo and Wade [18] extended the results of Grill [14] by studying the asymptotic behavior of (G_n). More precisely, let $S_n = X_1 + \cdots + X_n$ where the increments (X_n) are independent and

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identically distributed square integrable random vectors of $\mathbb{R}^d$ with mean μ and covariance matrix Γ. They proved the strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} G_n = \frac{1}{2}\mu \qquad \text{a.s.} \tag{1.2}$$

together with the asymptotic normality,

$$\frac{1}{\sqrt{n}}\Big(G_n - \frac{n}{2}\mu\Big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3}\Gamma\Big). \tag{1.3}$$

Curiously, no other references are available on the asymptotic behavior of the center of mass. The proofs of many results on G_n rely on independence and exchangeability of the increments of the walk. For example, one can observe that

$$G_n = \frac{1}{n}\sum_{k=1}^{n} S_k = \frac{1}{n}\sum_{k=1}^{n}\big(n-k+1\big)X_k \tag{1.4}$$

shares the same distribution as

$$\Sigma_n = \frac{1}{n}\sum_{k=1}^{n} kX_k.$$

A natural question concerns the asymptotic behavior of G_n in other situations where the increments of the walk are not independent and not identically distributed. Moreover, geometrical features of the random walk paths of (S_n) is a subject of ongoing interest. For example, the convex hull $C_n = \text{Conv}(S_1, \ldots, S_n)$ of the n first steps of (S_n) have recently received renewed attention [16,26]. More particularly, for the random walk in $\mathbb{R}^2$, the strong law of large numbers and the asymptotic normality of the perimeter and the diameter of C_n were established in [19,27].

In this paper, we investigate the asymptotic behavior of the center of mass of the multi-dimensional elephant random walk. It is a fascinating discrete-time random walk on $\mathbb{Z}^d$ where $d \geq 1$, which has a complete memory of its whole history. The increments depend on all the past of the walk and they are not exchangeable. The elephant random walk (ERW) was introduced by Schütz and Trimper [21] in the early 2000s, in order to investigate how long-range memory affects the random walk and induces a crossover from a diffusive to superdiffusive behavior. It was referred to as the ERW in allusion to the traditional saying that elephants can always remember where they have been before. The elephant starts at the origin at time zero, $S_0 = 0$. At time $n = 1$, the elephant moves in one of the $2d$ directions with the same probability $1/(2d)$. Afterwards, at time $n + 1$, the elephant chooses uniformly at random an integer k among the previous times $1, \ldots, n$. Then, it moves exactly in the same direction as that of time k with probability p or in one of the $2d - 1$ remaining directions with the same probability $(1 - p)/(2d - 1)$, where the parameter $p \in [0, 1]$ stands for the memory parameter of the ERW [4]. One can observe that the special case $p = 1/(2d)$ reduces back to the simple random walk on $\mathbb{Z}^d$. Therefore, the position of the elephant at time $n + 1$ is given by

$$S_{n+1} = S_n + X_{n+1} \tag{1.5}$$

where X_{n+1} is the $(n + 1)$th increment of the random walk. The ERW shows three different regimes depending on the location of its memory parameter p with respect to the critical value p_d lying between $1/(2d)$ and $3/4$,

$$p_d = \frac{2d+1}{4d}. \tag{1.6}$$

A wide literature is now available on the ERW in dimension $d = 1$ where $p_d = 3/4$. A strong law of large numbers and a central limit theorem for the position S_n, properly normalized, were established in the diffusive regime $p < 3/4$ and the critical regime $p = 3/4$, see [1,9,10,21] and the recent contributions [3,6,7,11,13,20,25]. The superdiffusive regime $p > 3/4$ is much harder to handle. Bercu [2] proved that the limit of the position of the ERW is not Gaussian. Quite recently, Kubota and Takei [17] showed that the fluctuation of the ERW around its limit in the superdiffusive regime is Gaussian. Finally, Bercu and Laulin in [4] extended all the results of [2] to the multi-dimensional ERW where $d \geq 1$.

Our strategy for proving asymptotic results for the center of mass of the elephant random walk (CMERW) is as follows. On the one hand, the behavior of position S_n is closely related to that of the sequence (M_n) defined, for all $n \geq 0$, by $M_n = a_n S_n$ with $a_1 = 1$ and, for all $n \geq 2$,

$$a_n = \prod_{k=1}^{n-1}\Big(\frac{k}{k+a}\Big) = \frac{\Gamma(a+1)\Gamma(n)}{\Gamma(n+a)} \tag{1.7}$$

where Γ stands for the Euler Gamma function and a is the fundamental parameter of the ERW defined by

$$a = \frac{2dp-1}{2d-1}. \tag{1.8}$$

We assume throughout all the paper that $a > -1$ inasmuch as $a = -1$ only appears in the singular case where $d = 1$ and $p = 0$. It follows from the very definition of the ERW that at any time $n \geq 1$,

$$\mathbb{E}\,[X_{n+1}|\mathcal{F}_n] = \Big(pI_d - \frac{1-p}{2d-1}I_d\Big)\frac{1}{n}\sum_{k=1}^{n} X_k = \frac{1}{n}\Big(\frac{2dp-1}{2d-1}\Big)S_n = \frac{a}{n}S_n \qquad \text{a.s.} \tag{1.9}$$

Hence, we obtain from (1.5) and (1.9) that for any $n \geq 1$,

$$\mathbb{E}\,[S_{n+1}|\mathcal{F}_n] = \Big(1 + \frac{a}{n}\Big)S_n \qquad \text{a.s.} \tag{1.10}$$

Therefore, we deduce from (1.7) and (1.10) that

$$\mathbb{E}\,[M_{n+1}|\mathcal{F}_n] = a_{n+1}\Big(1 + \frac{a}{n}\Big)S_n = a_n S_n = M_n \qquad \text{a.s.} \tag{1.11}$$

It means that (M_n) is a locally square-integrable martingale adapted to the filtration $(\mathcal{F}_n)$ where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. It can be rewritten [4, page 7] in the additive form

$$M_n = \sum_{k=1}^{n} a_k \varepsilon_k \tag{1.12}$$

where $\varepsilon_1 = S_1$ and, for all $n \geq 2$,

$$\varepsilon_n = S_n - \Big(\frac{a_{n-1}}{a_n}\Big)S_{n-1} = S_n - \Big(1 + \frac{a}{n-1}\Big)S_{n-1}. \tag{1.13}$$

On the other hand, an analogue of Eq. (1.4) is given by

$$\begin{aligned} G_n &= \frac{1}{n}\sum_{k=1}^{n} S_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}M_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\sum_{\ell=1}^{k} a_\ell\varepsilon_\ell = \frac{1}{n}\sum_{k=1}^{n} a_k\varepsilon_k\sum_{\ell=k}^{n}\frac{1}{a_\ell}, \\ &= \frac{1}{n}\sum_{k=1}^{n} a_k(b_n - b_{k-1})\varepsilon_k \end{aligned} \tag{1.14}$$

where the sequence (b_n) is given by $b_0 = 0$ and, for all $n \geq 1$,

$$b_n = \sum_{k=1}^{n} \frac{1}{a_k}. \tag{1.15}$$

In the particular case of the simple random walk on $\mathbb{Z}^d$, one can notice that $a = 0$ so $a_n = 1/n$ and $b_n = n(n+1)/2$. Hereafter, denoting

$$N_n = \sum_{k=1}^{n} a_k b_{k-1} \varepsilon_k, \tag{1.16}$$

it is straightforward to see that $\mathbb{E}[N_{n+1}|\mathcal{F}_n] = N_n$ a.s. since $\mathbb{E}[\varepsilon_{n+1}|\mathcal{F}_n] = 0$. Hence, (N_n) is also a locally square-integrable martingale adapted to the filtration $(\mathcal{F}_n)$. We deduce from (1.14) that

$$G_n = \frac{1}{n}(b_n M_n - N_n). \tag{1.17}$$

Relation (1.17) allows us to establish the asymptotic behavior of the CMERW via an extensive use of the strong law of large numbers and the central limit theorem for multi-dimensional martingales [8,12,15,23].

The paper is organized as follows. The main results are given in Section 2. We first investigate the diffusive regime $p < p_d$ and we establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for the CMERW. The asymptotic normality of the CMERW, properly normalized, is also provided. Next, we prove similar results in the critical regime $p = p_d$. Finally, we establish a strong limit theorem in the superdiffusive regime $p > p_d$. Our martingale approach is described in Section 3 while all technical proofs are postponed to Appendices A–C.

2. Main results

2.1. The diffusive regime

Our first result deals with the strong law of large numbers for the CMERW in the diffusive regime where $0 \leq p < p_d$. The following strong law for the CMERW will be deduced as a simple consequence of the strong law for (S_n). We recall that we assume $a > -1$ as $a = -1$ only occurs in the singular case where $d = 1$ and $p = 0$.

Theorem 2.1. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{n} G_n = 0 \qquad a.s. \tag{2.1}$$

More precisely, for any $\alpha > 1/2$,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} G_n = 0 \qquad a.s. \tag{2.2}$$

The almost sure rates of convergence for CMERW are as follows.

Theorem 2.2. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2} G_k G_k^T = \frac{2}{3(1-2a)(2-a)d} I_d \qquad a.s. \tag{2.3}$$

where I_d stands for the identity matrix of order d. In particular,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|G_k\|^2}{k^2} = \frac{2}{3(1-2a)(2-a)} \qquad \text{a.s.} \tag{2.4}$$

Moreover, we also have the upper-bound in the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|G_n\|^2}{2n \log\log n} \leq \frac{\big(\sqrt{3}+\sqrt{1-2a}\big)^2}{3(a+1)^2(1-2a)d} \qquad \text{a.s.} \tag{2.5}$$

Remark 2.1. The law of iterated logarithm for G_n involves the difference between $b_n M_n$ and N_n both satisfying a law of iterated logarithm with the same speed. It is not possible to make the addition of the two limits superior. Consequently, we were only able to obtain an upper-bound in (2.5). However, it is worth saying that for the simple random walk on $\mathbb{Z}^d$ for which $a = 0$, Strassen's invariance principle [24] implies that

$$\limsup_{n\to\infty} \frac{\|G_n\|^2}{2n \log\log n} = \frac{1}{3d} \qquad \text{a.s.}$$

which is not very far from the upper-bound $(1+\sqrt{3})^2/(3d)$ given in (2.5).

We are now interested in the asymptotic normality of the CMERW.

Theorem 2.3. *We have the joint asymptotic normality*

$$\frac{1}{\sqrt{n}} \begin{pmatrix} S_n \\ G_n \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}\big(0, \Gamma_d\big) \tag{2.6}$$

where Γ_d stands for the covariance matrix

$$\Gamma_d = \frac{1}{(1-2a)d} \begin{pmatrix} 1 & \frac{1}{(2-a)} \\ \frac{1}{(2-a)} & \frac{2}{3(2-a)} \end{pmatrix} \otimes I_d.$$

In particular,

$$\frac{1}{\sqrt{n}} G_n \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{2}{3(1-2a)(2-a)d} I_d\Big). \tag{2.7}$$

Remark 2.2. One can observe from Theorem 3.3 in [4] that the ratio of the asymptotic variances between the CMERW and the ERW is given by

$$R(a) = \frac{2}{3(2-a)}.$$

In the diffusive regime, this ratio lies between $2/9$ and $4/9$ and it is always smaller than 1, as one can see in Fig. 1. Moreover, in the special case where the elephant moves in one of the $2d$ directions with the same probability $p = 1/(2d) < p_d$, it follows from (1.8) that the fundamental parameter $a = 0$. Consequently, we deduce from (2.7) that

$$\frac{1}{\sqrt{n}} G_n \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3d} I_d\Big).$$

We find again the asymptotic normality (1.3) where the mean value $\mu = 0$ and the covariance matrix $\Gamma = \frac{1}{d} I_d$. An alternative approach to prove the asymptotic normality (2.7) via a

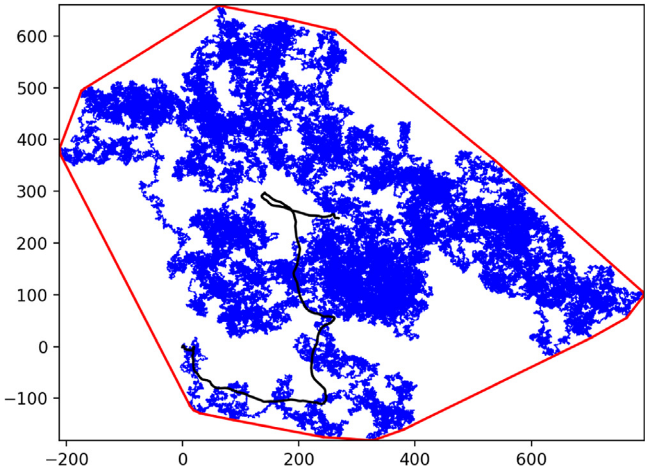

Fig. 1. The 2-dimensional ERW in blue, the CMERW in black and the convex hull in red, for $n = 10^6$ steps and a diffusive memory parameter $p = 1/2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

functional central limit theorem for the multi-dimensional ERW can be found in [5]. One can observe from Theorem 1 in [1] that (2.7) also holds in the singular case $a = -1$ which only appears when $d = 1$ and $p = 0$.

2.2. The critical regime

Hereafter, we investigate the critical regime where the memory parameter $p = p_d$.

Theorem 2.4. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}\log n} G_n = 0 \qquad a.s. \tag{2.8}$$

More precisely, for any $\alpha > 1/2$,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}(\log n)^{\alpha}} G_n = 0 \qquad a.s. \tag{2.9}$$

The almost sure rates of convergence for the CMERW are as follows.

Theorem 2.5. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{1}{(k\log k)^2} G_k G_k^T = \frac{4}{9d} I_d \qquad a.s. \tag{2.10}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{\|G_k\|^2}{(k\log k)^2} = \frac{4}{9} \qquad a.s. \tag{2.11}$$

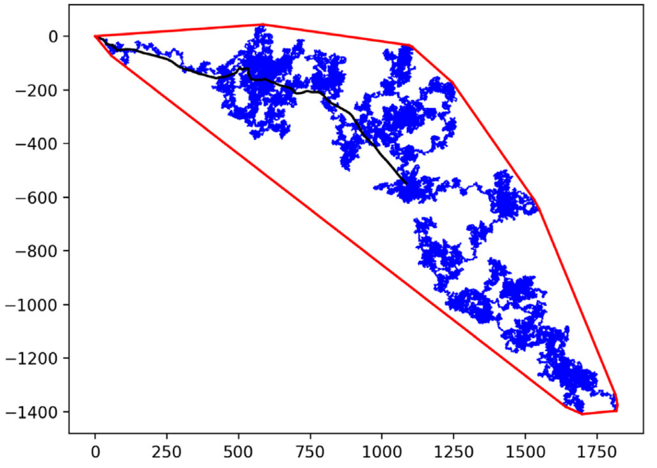

Fig. 2. The 2-dimensional ERW in blue, the CMERW in black and the convex hull in red, for $n = 10^6$ steps and a critical memory parameter $p = 5/8$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Moreover, we also have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|G_n\|^2}{2n \log n \log\log\log n} = \frac{4}{9d} \qquad a.s. \tag{2.12}$$

Our next result concerns the asymptotic normality of the CMERW.

Theorem 2.6. *We have the joint asymptotic normality*

$$\frac{1}{\sqrt{n}} \begin{pmatrix} S_n \\ G_n \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma_d) \tag{2.13}$$

where Σ_d stands for the covariance matrix

$$\Sigma_d = \frac{1}{d} \begin{pmatrix} 1 & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{9} \end{pmatrix} \otimes I_d.$$

In particular,

$$\frac{1}{\sqrt{n \log n}} G_n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{4}{9d} I_d\right). \tag{2.14}$$

Remark 2.3. In the critical regime, the ratio of the asymptotic variances between the CMERW and the ERW is 4/9. See Fig. 2 for a trajectory of the ERW and the CMERW in the diffusive regime.

2.3. The superdiffusive regime

Finally, we focus our attention on the superdiffusive regime where $p > p_d$. The almost sure convergence of (S_n), properly normalized by n^a, yields the following strong limit theorem for the CMERW.

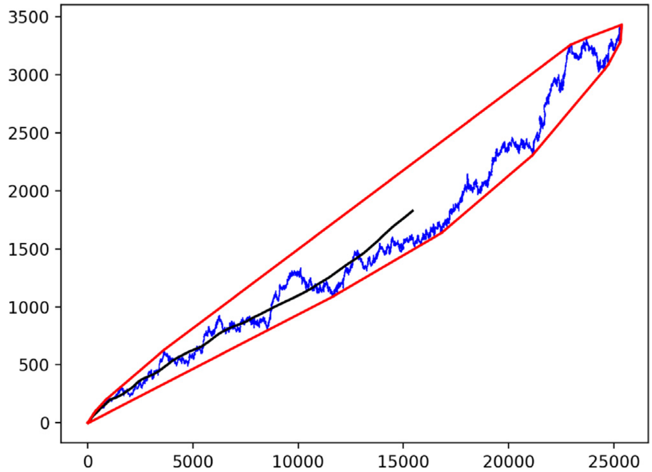

Fig. 3. The 2-dimensional ERW in blue, the CMERW in black and the convex hull in red, for $n = 10^6$ steps and a superdiffusive memory parameter $p = 3/4$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Theorem 2.7. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{n^a} G_n = G \qquad a.s. \tag{2.15}$$

where the limiting value G is a non-degenerate random vector of $\mathbb{R}^d$. Moreover, we also have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big\| \frac{1}{n^a} G_n - G \Big\|^2\Big] = 0. \tag{2.16}$$

Remark 2.4. The expected value of G is zero and its covariance matrix is given by

$$\mathbb{E}\big[GG^T\big] = \frac{1}{d(a+1)^2(2a-1)^2\Gamma(2a-1)} I_d.$$

The distribution of G is far from being known. See Fig. 3 for a trajectory of the ERW and the CMERW in the superdiffusive regime.

3. A multi-dimensional martingale approach

We already saw from (1.17) that the CMERW can be rewritten as

$$G_n = \frac{1}{n}(b_n M_n - N_n).$$

In order to investigate the asymptotic behavior of (G_n), we introduce the multi-dimensional martingale $(\mathcal{M}_n)$ defined by

$$\mathcal{M}_n = \begin{pmatrix} M_n \\ N_n \end{pmatrix} \tag{3.1}$$

where (M_n) and (N_n) are the two locally square-integrable martingales given by (1.12) and (1.16). The main difficulty we face here is that the predictable quadratic variation of (M_n)

and (N_n) increase to infinity with two different speeds. A matrix normalization is necessary to establish the asymptotic behavior of the CMERW. Let (V_n) be the sequence of positive definite diagonal matrices of order $2d$ given by

$$V_n = \frac{1}{n\sqrt{n}} \begin{pmatrix} b_n & 0 \\ 0 & 1 \end{pmatrix} \otimes I_d \tag{3.2}$$

where $A \otimes B$ stands for the Kronecker product of the matrices A and B.

Lemma 3.1. *The sequence $(\mathcal{M}_n)$ is a locally square-integrable martingale of $\mathbb{R}^{2d}$. Its predictable quadratic variation $\langle \mathcal{M} \rangle_n$ satisfies in the diffusive regime where $a < 1/2$,*

$$\lim_{n\to\infty} V_n \langle \mathcal{M} \rangle_n V_n^T = V \qquad a.s. \tag{3.3}$$

where the limiting matrix V is given by

$$V = \frac{1}{d(a+1)^2} \begin{pmatrix} \frac{1}{1-2a} & \frac{1}{2-a} \\ \frac{1}{2-a} & \frac{1}{3} \end{pmatrix} \otimes I_d. \tag{3.4}$$

Remark 3.1. Via the same lines as in the proof of Lemma 3.1, we find that in the critical regime $a = 1/2$, the sequence of normalization matrices (V_n) has to be replaced by

$$W_n = \frac{1}{n\sqrt{n \log n}} \begin{pmatrix} b_n & 0 \\ 0 & 1 \end{pmatrix} \otimes I_d. \tag{3.5}$$

Moreover, the limiting matrix in (3.3) must be changed by

$$W = \frac{4}{9d} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I_d. \tag{3.6}$$

Proof. The increments of the ERW are bounded by 1, that is for any time $n \geq 1$, $\|X_n\| = 1$. Hence, it follows from (1.5) that $\|S_n\| \leq n$ and $\|G_n\| \leq n$ which imply that $\|M_n\| \leq na_n$ and $\|N_n\| \leq na_nb_n + n^2$. We already saw in Section 1 that $(\mathcal{M}_n)$ is a locally square-integrable martingale. Denote $\Delta M_n = M_n - M_{n-1}$, and similarly for other processes. It follows from (1.12), (1.13) and (1.16) that the predictable quadratic variation associated with $(\mathcal{M}_n)$ is the square matrix of order $2d$ given, for all $n \geq 1$, by

$$\langle \mathcal{M} \rangle_n = \sum_{k=1}^{n} \mathbb{E}\left[\begin{pmatrix} \Delta M_k \\ \Delta N_k \end{pmatrix} \begin{pmatrix} \Delta M_k \\ \Delta N_k \end{pmatrix}^T \Big| \mathcal{F}_{k-1} \right] = \sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes \mathbb{E}\big[\varepsilon_k \varepsilon_k^T \big| \mathcal{F}_{k-1}\big]. \tag{3.7}$$

Moreover, we deduce from formulas (A.7) and (B.3) in [4] that for all $n \geq 1$,

$$\mathbb{E}\left[\varepsilon_{n+1}\varepsilon_{n+1}^T | \mathcal{F}_n\right] = \frac{1}{d} I_d + a\Big(\frac{1}{n}\Sigma_n - \frac{1}{d} I_d\Big) - \Big(\frac{a}{n}\Big)^2 S_n S_n^T \qquad \text{a.s.} \tag{3.8}$$

where Σ_n is a random positive definite matrix of order d satisfying

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_n = \frac{1}{d} I_d \qquad \text{a.s.} \tag{3.9}$$

Consequently, we obtain from (3.7) together with (3.8) that

$$\langle \mathcal{M} \rangle_n = \frac{1}{d} \sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes I_d + a \sum_{k=1}^{n-1} a_{k+1}^2 \begin{pmatrix} 1 & b_k \\ b_k & b_k^2 \end{pmatrix} \otimes \Big(\frac{1}{k}\Sigma_k - \frac{1}{d} I_d\Big) - \xi_n \tag{3.10}$$

where

$$\xi_n = a^2 \sum_{k=1}^{n-1} \Big(\frac{a_{k+1}}{k}\Big)^2 \begin{pmatrix} 1 & b_k \\ b_k & b_k^2 \end{pmatrix} \otimes S_k S_k^T.$$

According to Theorem 3.1 in [4], the remainder ξ_n plays a negligible role as

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \qquad \text{a.s.} \tag{3.11}$$

Hereafter, it is not hard to see that

$$V_n \Big(\sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes I_d\Big) V_n^T = \frac{1}{n^3} \begin{pmatrix} b_n^2 \sum_{k=1}^n a_k^2 & b_n \sum_{k=1}^n a_k^2 b_{k-1} \\ b_n \sum_{k=1}^n a_k^2 b_{k-1} & \sum_{k=1}^n a_k^2 b_{k-1}^2 \end{pmatrix} \otimes I_d.$$

Furthermore, from a well-known property of the Euler Gamma function, we have

$$\lim_{n\to\infty} \frac{\Gamma(n+a)}{\Gamma(n) n^a} = 1. \tag{3.12}$$

Hence, we obtain from (1.7), (1.15) and (3.12) that

$$\lim_{n\to\infty} n^a a_n = \Gamma(a+1) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n}{n^{a+1}} = \frac{1}{\Gamma(a+2)}. \tag{3.13}$$

Consequently, as soon as $a < 1/2$, we immediately find from (3.13) that

$$\lim_{n\to\infty} \frac{b_n^2}{n^3} \sum_{k=1}^n a_k^2 = \frac{1}{(1-2a)(a+1)^2},$$

$$\lim_{n\to\infty} \frac{b_n}{n^3} \sum_{k=1}^n a_k^2 b_{k-1} = \frac{1}{(2-a)(a+1)^2},$$

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^n a_k^2 b_{k-1}^2 = \frac{1}{3(a+1)^2}.$$

Therefore,

$$\lim_{n\to\infty} V_n \Big(\sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes I_d\Big) V_n^T = \frac{1}{(a+1)^2} \begin{pmatrix} \frac{1}{1-2a} & \frac{1}{2-a} \\ \frac{1}{2-a} & \frac{1}{3} \end{pmatrix} \otimes I_d. \tag{3.14}$$

Finally, it follows from the combination of (3.9), (3.10), (3.11) and (3.14) that

$$\lim_{n\to\infty} V_n \langle \mathcal{M} \rangle_n V_n^T = \frac{1}{d(a+1)^2} \begin{pmatrix} \frac{1}{1-2a} & \frac{1}{2-a} \\ \frac{1}{2-a} & \frac{1}{3} \end{pmatrix} \otimes I_d. \qquad \text{a.s.} \tag{3.15}$$

which is exactly what we wanted to prove. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Two non-standard results on martingales

The proofs of our main results rely on two non-standard central limit theorem and quadratic strong law for multi-dimensional martingales. A simplified version of Theorem 1 of Touati [23] is as follows.

Theorem A.1. *Let $(\mathcal{M}_n)$ be a locally square-integrable martingale of $\mathbb{R}^\delta$ adapted to a filtration $(\mathcal{F}_n)$, with predictable quadratic variation $\langle \mathcal{M} \rangle_n$. Let (V_n) be a sequence of non-random square matrices of order δ such that $\|V_n\|$ decreases to 0 as n goes to infinity. Assume that there exists a symmetric and positive semi-definite matrix V such that*

$$V_n \langle \mathcal{M} \rangle_n V_n^T \xrightarrow[n\to\infty]{\mathbb{P}} V. \tag{H.1}$$

Moreover, assume that Lindeberg's condition is satisfied, that is for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta\mathcal{M}_k\|^2 \mathrm{I}_{\{\|V_n \Delta\mathcal{M}_k\| > \varepsilon\}} \big| \mathcal{F}_{k-1}\big] \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{H.2}$$

where $\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$. Then, we have the asymptotic normality

$$V_n \mathcal{M}_n \xrightarrow{\mathcal{L}} \mathcal{N}\big(0, V\big). \tag{A.1}$$

The quadratic strong law requires more restrictive assumptions. The following result is a simplified version of Theorem 2.1 of Chaabane and Maaouia [8] where the normalization matrices (V_n) are diagonal.

Theorem A.2. *Let $(\mathcal{M}_n)$ be a locally square-integrable martingale of $\mathbb{R}^\delta$ adapted to a filtration $(\mathcal{F}_n)$, with predictable quadratic variation $\langle \mathcal{M} \rangle_n$. Let (V_n) be a sequence of non-random positive definite diagonal matrices of order δ such that its diagonal terms decrease to zero at polynomial rates. Assume that (H.1) and (H.2) hold almost surely. Moreover, suppose that there exists $\beta \in]1, 2]$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{\big(\log(\det V_n^{-1})^2\big)^\beta} \mathbb{E}\big[\|V_n \Delta\mathcal{M}_n\|^{2\beta} \big| \mathcal{F}_{n-1}\big] < \infty \qquad a.s. \tag{H.3}$$

Then, we have the quadratic strong law

$$\lim_{n\to\infty} \frac{1}{\log(\det V_n^{-1})^2} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) V_k \mathcal{M}_k \mathcal{M}_k^T V_k^T = V \qquad a.s. \tag{A.2}$$

Appendix B. Proofs of the almost sure convergence results

B.1. The diffusive regime

Proof of Theorem 2.1. We already saw from Theorem 3.1 in [4] that

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \qquad \text{a.s.} \tag{B.1}$$

Consequently, the almost sure convergence (2.1) immediately follows from (B.1) together with the Toeplitz lemma given e.g. by Lemma 2.2.13 in [12]. Moreover, we also have from Remark 3.1 in [4] that for any $\alpha > 1/2$,

$$\lim_{n\to\infty} \frac{S_n}{n^\alpha} = 0 \qquad \text{a.s.}$$

Hence, we obtain (2.2) using once again Toeplitz lemma. □

Proof of Theorem 2.2. Our goal is to check that all the hypotheses of Theorem A.2 are satisfied. Thanks to Lemma 3.1, hypothesis (H.1) holds almost surely. In order to verify that Lindeberg's condition (H.2) is satisfied, we have from (3.1) together with (1.12), (1.16) and V_n given by (3.2) that for all $1 \le k \le n$

$$V_n \Delta\mathcal{M}_k = \frac{a_k}{n\sqrt{n}} \begin{pmatrix} b_n \varepsilon_k \\ b_{k-1}\varepsilon_k \end{pmatrix},$$

which implies that

$$\| V_n \Delta\mathcal{M}_k \|^2 \le \frac{2a_k^2 b_n^2}{n^3} \|\varepsilon_k\|^2. \tag{B.2}$$

Consequently, we obtain that for all $\varepsilon > 0$,

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{E}\big[\| V_n \Delta\mathcal{M}_k \|^2 \mathrm{I}_{\{\| V_n \Delta\mathcal{M}_k \| > \varepsilon\}} \big| \mathcal{F}_{k-1}\big] &\le \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \mathbb{E}\big[\| V_n \Delta\mathcal{M}_k \|^4 \big| \mathcal{F}_{k-1}\big], \\ &\le \frac{4b_n^4}{\varepsilon^2 n^6} \sum_{k=1}^{n} a_k^4 \mathbb{E}\big[\|\varepsilon_k\|^4 \big| \mathcal{F}_{k-1}\big], \\ &\le \frac{4b_n^4}{\varepsilon^2 n^6} \sup_{1\le k\le n} \mathbb{E}\big[\|\varepsilon_k\|^4 \big| \mathcal{F}_{k-1}\big] \sum_{k=1}^{n} a_k^4. \end{aligned} \tag{B.3}$$

However, it follows from the right-hand side of formula (4.11) in [4] that

$$\sup_{1\le k\le n} \mathbb{E}\big[\|\varepsilon_k\|^4 \big| \mathcal{F}_{k-1}\big] \le \frac{4}{3} \qquad \text{a.s.} \tag{B.4}$$

Therefore, we infer from (B.3) that for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \mathbb{E}\big[\| V_n \Delta\mathcal{M}_k \|^2 \mathrm{I}_{\{\| V_n \Delta\mathcal{M}_k \| > \varepsilon\}} \big| \mathcal{F}_{k-1}\big] \le \frac{16\, b_n^4}{3\,\varepsilon^2 n^6} \sum_{k=1}^{n} a_k^4 \qquad \text{a.s.} \tag{B.5}$$

Moreover, we have from (3.13) that

$$b_n^4 \sum_{k=1}^{n} a_k^4 = O(n^5). \tag{B.6}$$

Consequently, (B.5) together with (B.6) ensure that Lindeberg's condition (H.2) holds almost surely, that is for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\big[\| V_n \Delta\mathcal{M}_k \|^2 \mathrm{I}_{\{\| V_n \Delta\mathcal{M}_k \| > \varepsilon\}} \big| \mathcal{F}_{k-1}\big] = 0 \qquad \text{a.s.} \tag{B.7}$$

We will now check that condition (H.3) is satisfied in the special case $\beta = 2$, that is

$$\sum_{n=1}^{\infty} \frac{1}{\big(\log(\det V_n^{-1})^2\big)^2} \mathbb{E}\big[\|V_n \Delta\mathcal{M}_n\|^4 \big| \mathcal{F}_{n-1}\big] < \infty \qquad \text{a.s.} \tag{B.8}$$

We have from (3.2) that

$$\det V_n^{-1} = \Big(\frac{n^{3/2}}{b_n}\Big)^d. \tag{B.9}$$

Hence, we find from (3.13) and (B.9) that

$$\lim_{n\to\infty} \frac{\log(\det V_n^{-1})^2}{\log n} = d(1-2a). \tag{B.10}$$

Consequently, we can replace $\log(\det V_n^{-1})^2$ by $\log n$ in (B.8). Hereafter, we obtain from (B.2) and (B.4) that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{(\log n)^2} \mathbb{E}\big[\|V_n \Delta\mathcal{M}_n\|^4 \big| \mathcal{F}_{n-1}\big] &= O\Big(\sum_{n=1}^{\infty} \frac{1}{(\log n)^2} \frac{a_n^4 b_n^4}{n^6} \mathbb{E}\big[\|\varepsilon_n\|^4 \big| \mathcal{F}_{n-1}\big]\Big),\\ &= O\Big(\sum_{n=1}^{\infty} \frac{1}{(\log n)^2} \frac{a_n^4 b_n^4}{n^6}\Big).\end{aligned} \tag{B.11}$$

However, we have from (3.13) that

$$\lim_{n\to\infty} \frac{a_n^4 b_n^4}{n^4} = \frac{1}{(a+1)^4}. \tag{B.12}$$

Therefore, (B.11) together with (B.12) immediately leads to (B.8). We are now in a position to apply the quadratic strong law given by Theorem A.2. We have from (A.2) and (B.10) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) V_k \mathcal{M}_k \mathcal{M}_k^T V_k^T = d(1-2a)V \qquad \text{a.s.} \tag{B.13}$$

where the limiting matrix V is given by (3.4). However, it follows from (1.14), (3.1) and (3.2) that

$$\frac{1}{\sqrt{n}} G_n = v^T V_n \mathcal{M}_n \qquad \text{where} \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes I_d. \tag{B.14}$$

Consequently, we deduce from (B.13) and (B.14) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) \frac{1}{k} G_k G_k^T = d(1-2a) v^T V v \qquad \text{a.s.} \tag{B.15}$$

Furthermore, we obtain from (3.13) and (B.9) that

$$\lim_{n\to\infty} n\Big(\frac{(\det V_n)^2 - (\det V_{n+1})^2}{(\det V_n)^2}\Big) = d(1-2a).$$

Hence, (B.15) clearly leads to convergence (2.3),

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2} G_k G_k^T = v^T V v = \frac{2}{3(1-2a)(2-a)d} I_d \qquad \text{a.s.} \tag{B.16}$$

By taking the trace on both sides of (B.16), we also obtain that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|G_k\|^2}{k^2} = \frac{2}{3(1-2a)(2-a)} \qquad \text{a.s.} \tag{B.17}$$

Finally, we shall proceed to the proof of the upper-bound (2.5) in the law of iterated logarithm. Denote

$$\tau_n = \sum_{k=1}^{n} a_k^2 b_{k-1}^2. \tag{B.18}$$

We already saw from (B.12) that $a_n^4 b_{n-1}^4 \tau_n^{-2}$ is equivalent to $9n^{-2}$. It implies that

$$\sum_{n=1}^{+\infty} \frac{a_n^4 b_{n-1}^4}{\tau_n^2} < \infty. \tag{B.19}$$

Moreover, we have from (3.9), (3.10), (3.11) and (B.18) that

$$\lim_{n\to\infty} \frac{1}{\tau_n} \langle N \rangle_n = \frac{1}{d} I_d \qquad \text{a.s.}$$

Consequently, we deduce from the law of iterated logarithm for martingales due to Stout [22], see also Corollary 6.4.25 in [12], that (N_n) satisfies for any vector $u \in \mathbb{R}^d$,

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2\tau_n \log\log \tau_n}\Big)^{1/2} \langle u, N_n\rangle &= -\liminf_{n\to\infty} \Big(\frac{1}{2\tau_n \log\log \tau_n}\Big)^{1/2} \langle u, N_n\rangle \\ &= \frac{1}{\sqrt{d}} \|u\| \qquad \text{a.s.} \end{aligned} \tag{B.20}$$

However, since τ_n is equivalent to $n^3/3(a+1)^2$, (B.20) immediately lead to

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, N_n\rangle &= -\liminf_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, N_n\rangle \\ &= \frac{1}{\sqrt{3d}(a+1)} \|u\| \qquad \text{a.s.} \end{aligned} \tag{B.21}$$

Furthermore, it was already shown by formula (5.17) in [4] that for any vector $u \in \mathbb{R}^d$,

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{n^{2a}}{2n \log\log n}\Big)^{1/2} \langle u, M_n\rangle &= -\liminf_{n\to\infty} \Big(\frac{n^{2a}}{2n \log\log n}\Big)^{1/2} \langle u, M_n\rangle \\ &= \frac{\Gamma(a+1)}{\sqrt{d(1-2a)}} \|u\| \qquad \text{a.s.} \end{aligned} \tag{B.22}$$

Therefore, we deduce from (1.17) and (3.13) together with (B.21) and (B.22) that for any vector u of $\mathbb{R}^d$,

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \langle u, G_n\rangle &= \limsup_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, b_n M_n - N_n\rangle \\ &\leq \limsup_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, b_n M_n\rangle + \limsup_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, -N_n\rangle \\ &\leq \limsup_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, b_n M_n\rangle - \liminf_{n\to\infty} \Big(\frac{1}{2n \log\log n}\Big)^{1/2} \frac{1}{n} \langle u, N_n\rangle \\ &\leq \frac{\|u\|}{\sqrt{d}(a+1)} \Big(\frac{1}{\sqrt{1-2a}} + \frac{1}{\sqrt{3}}\Big) \qquad \text{a.s.} \end{aligned} \tag{B.23}$$

By the same token, we also find that for any vector u of $\mathbb{R}^d$,

$$\liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\langle u, G_n\rangle = \liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, b_n M_n - N_n\rangle$$
$$\geq -\frac{\|u\|}{\sqrt{d}(a+1)}\Big(\frac{1}{\sqrt{1-2a}}+\frac{1}{\sqrt{3}}\Big) \qquad \text{a.s.} \tag{B.24}$$

Consequently, we obtain from (B.23) and (B.24) that for any vector u of $\mathbb{R}^d$,

$$\limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)\langle u, G_n\rangle^2 \leq \frac{\|u\|^2}{d(a+1)^2}\Big(\frac{1}{\sqrt{1-2a}}+\frac{1}{\sqrt{3}}\Big)^2 \qquad \text{a.s.} \tag{B.25}$$

One can observe that the upper-bound in (B.25) is close to the optimal bound

$$(vu)^T V vu = \frac{2\|u\|^2}{3(1-2a)(2-a)d}.$$

Finally, by taking all rational points on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the bound in (B.25) holds simultaneously for all of them, which implies that

$$\limsup_{n\to\infty}\frac{\|G_n\|^2}{2n\log\log n} \leq \sup_{u\in\mathbb{Q}^d\cap\mathbb{S}^{d-1}}\limsup_{n\to\infty}\frac{\langle u, G_n\rangle^2}{2n\log\log n} \leq \frac{\big(\sqrt{3}+\sqrt{1-2a}\big)^2}{3(a+1)^2(1-2a)d} \qquad \text{a.s.}$$

completing the proof of Theorem 2.2. □

B.2. The critical regime

Proof of Theorem 2.4. We have from Theorem 3.4 in [4] that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}\log n}S_n = 0 \qquad \text{a.s.} \tag{B.26}$$

Hence, (2.8) clearly follows from (B.26) together with the Toeplitz lemma. Moreover, we also have from Remark 3.3 in [4] that for any $\alpha > 1/2$,

$$\lim_{n\to\infty}\frac{S_n}{\sqrt{n}(\log n)^{\alpha}} = 0 \qquad \text{a.s.}$$

Hence, we obtain (2.9) using once again Toeplitz lemma. □

Proof of Theorem 2.5. The proof of the quadratic strong law (2.10) is left to the reader as it follows essentially the same lines as that of (2.3). The only minor change is that the matrix V_n has to be replaced by the matrix W_n defined in (3.6). We shall now proceed to the proof of the law of iterated logarithm given by (2.12). On the one hand, it follows from (B.21) with $a = 1/2$ that for any vector $u \in \mathbb{R}^d$,

$$\limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, N_n\rangle = -\liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, N_n\rangle$$
$$= \frac{2}{3\sqrt{3d}}\|u\| \qquad \text{a.s.}$$

which immediately leads to

$$\limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, N_n\rangle = 0 \qquad \text{a.s.}$$

On the other hand, we obtain from the law of iterated logarithm for S_n given in Theorem 3.5 of [4] that for any vector $u \in \mathbb{R}^d$,

$$\begin{aligned}
&\limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\langle u, G_n\rangle\\
&= \limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, b_n M_n - N_n\rangle\\
&= \limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, b_n M_n\rangle\\
&= \limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\frac{1}{n}\langle u, a_n b_n S_n\rangle\\
&= \limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\frac{2}{3}\langle u, S_n\rangle\\
&= -\liminf_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\frac{2}{3}\langle u, S_n\rangle\\
&= \frac{2}{3\sqrt{d}}\|u\| \qquad \text{a.s.}
\end{aligned} \tag{B.27}$$

Hence, we clearly deduce from (B.27) that for any vector $u \in \mathbb{R}^d$,

$$\limsup_{n\to\infty}\frac{1}{2n\log n\log\log\log n}\langle u, G_n\rangle^2 = \frac{4}{9d}\|u\|^2 \qquad \text{a.s.} \tag{B.28}$$

By taking all rational points on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the bound in (B.28) holds simultaneously for all of them, which implies that

$$\limsup_{n\to\infty}\frac{\|G_n\|^2}{2n\log n\log\log\log n} \leq \sup_{u\in\mathbb{Q}^d\cap\mathbb{S}^{d-1}}\limsup_{n\to\infty}\frac{\langle u, G_n\rangle^2}{2n\log n\log\log\log n} = \frac{4}{9d} \qquad \text{a.s.}$$

In addition, for any single $u \in \mathbb{S}^{d-1}$, we also obtain the reverse inequality

$$\limsup_{n\to\infty}\frac{\|G_n\|^2}{2n\log n\log\log\log n} \geq \limsup_{n\to\infty}\frac{\langle u, G_n\rangle^2}{2n\log n\log\log\log n} = \frac{4}{9d} \qquad \text{a.s.}$$

It immediately leads to (2.12) which achieves the proof of Theorem 2.5. □

B.3. The superdiffusive regime

Proof of Theorem 2.7. It follows from Theorem 3.7 in [4] that

$$\lim_{n\to\infty}\frac{1}{n^a}S_n = L \qquad \text{a.s.} \tag{B.29}$$

where the limiting value L is a non-degenerate random vector of $\mathbb{R}^d$. Hence, (B.29) together with the Toeplitz lemma imply (2.15) where the limiting value

$$G = \frac{1}{a+1}L.$$

Moreover, we have from (1.17) that

$$\mathbb{E}\Big[\Big\|\frac{1}{n^a}G_n - G\Big\|^2\Big] = \mathbb{E}\Big[\Big\|\frac{1}{n^{a+1}}(b_n M_n - N_n) - G\Big\|^2\Big],$$
$$\leq 2\mathbb{E}\Big[\Big\|\frac{a_n b_n}{n^{a+1}}S_n - G\Big\|^2\Big] + 2\mathbb{E}\Big[\Big\|\frac{1}{n^{a+1}}N_n\Big\|^2\Big].$$

On the one hand, we already saw from (3.13) that

$$\lim_{n\to\infty}\frac{a_n b_n}{n} = \frac{1}{a+1}.$$

Consequently, we deduce from the mean square convergence (3.12) in [4] that

$$\lim_{n\to\infty}\mathbb{E}\Big[\Big\|\frac{a_n b_n}{n^{a+1}}S_n - G\Big\|^2\Big] = 0. \tag{B.30}$$

On the other hand, $\mathbb{E}[\|N_n\|^2] = \mathbb{E}[\mathrm{Tr}\langle N_n\rangle] \leq \tau_n$ where τ_n is given by (B.18). Since τ_n is equivalent to $n^3/3(a+1)^2$ and $a > 1/2$, it is not hard to see that

$$\lim_{n\to\infty}\mathbb{E}\Big[\Big\|\frac{1}{n^{a+1}}N_n\Big\|^2\Big] = 0. \tag{B.31}$$

Finally, we obtain (2.16) from (B.30) and (B.31), completing the proof of Theorem 2.7. □

Appendix C. Proofs of the asymptotic normality results

C.1. The diffusive regime

Proof of Theorem 2.3. On the one hand, it follows from (B.14) that

$$\frac{1}{\sqrt{n}}\begin{pmatrix} S_n \\ G_n \end{pmatrix} = U_n^T V_n \mathcal{M}_n \qquad \text{where} \qquad U_n = \begin{pmatrix} u_n & 1 \\ 0 & -1 \end{pmatrix} \otimes I_d$$

with $u_n = n/a_n b_n$. On the other hand, we deduce from (3.3) and (B.7) that the two conditions (H.1) and (H.2) of Theorem A.1 are satisfied. In addition, (3.13) ensures that

$$\lim_{n\to\infty} U_n = U \qquad \text{where} \qquad U = \begin{pmatrix} a+1 & 1 \\ 0 & -1 \end{pmatrix} \otimes I_d.$$

Consequently, we obtain that

$$\frac{1}{\sqrt{n}}\begin{pmatrix} S_n \\ G_n \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma_d).$$

The asymptotic covariance matrix $\Gamma_d = U^T V U$ where V is given by (3.4). It clearly leads to (2.6) as

$$U^T V U = \frac{1}{(1-2a)d}\begin{pmatrix} 1 & \frac{1}{(2-a)} \\ \frac{1}{(2-a)} & \frac{2}{3(2-a)} \end{pmatrix} \otimes I_d. \quad \square$$

C.2. The critical regime

Proof of Theorem 2.6. The proof follows exactly the same lines as that of Theorem 2.3 replacing V_n by W_n. The details are left to the reader. □

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